Incentive Design and Pricing under Limited Inventory

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A firm faces random demand for a service it delivers on a given future date. To boost demand, the firm hires a sales agent who exerts unobservable effort continuously over time. The firm is concerned not only with increasing current demand, but also with smoothing demand over time to avoid losing goodwill if realized demand exceeds available inventory. We study the firm's incentive design problem using a novel continuous-time principal-agent framework, in which demand drifts over time in response to an agent's unobserved effort as well as the price the firm charges. To induce the agent's sales effort, the firm chooses an incentive scheme that depends on the remaining inventory and the time to the service (e.g., time to departure in the case of airlines). We characterize the firm's optimal incentive scheme under both static and dynamic pricing policies. Using parameters calibrated from the airline industry, we numerically show that under dynamic pricing, using a static incentive scheme helps the firm reap nearly all the benefits of the corresponding dynamic incentive scheme. Using a fully static strategy, on the other hand, results in a significant loss of efficiency. We also compare two partially dynamic strategies in which the firm uses dynamic pricing or dynamic contracting but not both. We show that when inventory levels are high and the demand is inelastic, the dynamic-contracting-only strategy outperforms the dynamic-pricing-only strategy; when inventory levels are low and the demand is elastic, however, the dynamic-pricing-only strategy outperforms the dynamic-contracting-only strategy. Finally, we analyze a case in which the firm operates on price segments and use our analytical framework to gain insight into the firm's choice of price level and length of each segment.

Key words: Incentive design, moral hazard, pricing, limited inventory, marketing-operations interface

1. Introduction

Firms across myriad service industries rely on sales agents to generate revenue. For example, in 2019, travel agencies remained the most popular sales channel for airline tickets, accounting for 44% of total sales; air ticket sales through agents were "up and rising" (Montevago 2019). The COVID-19 pandemic has proven the lasting value of travel agencies to customers and underlined their crucial role in the post-pandemic recovery of the industry (Kiesnoski 2021). The ongoing digital revolution has disrupted traditional business models and underscored the paramount importance of motivating sales efforts through the design of appropriate incentives. Yet, in the context of service industries where limited and perishable inventory is a hallmark, the problem of incentive design for sales agents has not been formally examined.

Two main reasons account for the lack of rigorous research on this important topic. First, generating sales entails matching supply with demand, which means supply must be adequate for demand to translate into sales. The salesforce compensation literature (e.g., Basu et al. 1985, Lal and Srinivasan 1993, Oyer 2000) focuses on designing incentive schemes to motivate a salesperson to exert effort that boosts the demand for a product; the salesperson's effort is unobservable to the principal, so moral hazard arises. A major limitation of the literature is that almost all of it assumes an *unlimited* inventory level, such that the entire demand can be fulfilled. Several recent papers (e.g., Dai and Jerath 2013, Dai, Ke, and Ryan 2021, Li, Chen, and Rong 2020, Song and Xiao 2021) overcome this limitation by imposing an inventory constraint but rely on a *single-period* model in which the process of matching supply with demand is treated as a one-shot interaction, which does not capture the intertemporal dynamics in the sales-generation process. Second, dynamic pricing is the norm for the airline industry and others facing limited and perishable service capacity. Yet, the salesforce compensation literature traditionally takes prices as static and exogenous. For these reasons, existing theoretical frameworks are not suitable for modeling the salesforce compensation problem in the service industry, such as airlines.

Methodology-wise, even a *single*-period moral hazard problem is challenging due to (1) its infinite-dimensional optimization nature (Dai, Ke, and Ryan 2021, p. 2215) and (2) the controversial validity of the commonly used first-order approach (Laffont and Martimort 2002, p. 200). A discrete-time moral hazard model, which extends a single-period model to multiple periods, is generally intractable and thus inappropriate for dynamic operations management problems (Plambeck and Zenios 2000, p. 240). Fortunately, continuous-time moral hazard models, as pioneered by Sannikov (2008), can characterize the general contract form without suffering from the curse of dimensionality and can decompose the dynamic principal-agent problem into a series of static problems. This makes continuous-time models more suitable for providing clear characterizations of optimal contracts and managerial insights that are not easily obtained from discrete-time principal-agent models.

In this paper, we develop a continuous-time modeling framework to account for (1) incentive design, (2) dynamic pricing, and (3) limited inventory. Our model reflects the view that customers do not necessarily arrive all at once and sales generation occurs continuously over time. In this continuous-time setting, the primary job of a salesperson is to boost demand, yet the desire of a customer-centric firm goes beyond clearing its supply immediately. Reasonable service availability and cumulative revenue over the whole selling period are important concerns for the firm. We seek to answer the following research questions: (1) Under static pricing, how should the firm design its incentive scheme? (2) Under dynamic pricing, how should the firm set its incentive scheme and price over time? (3) If the firm can only dynamically adjust either its incentive scheme or price but not both, which dimension should the firm prioritize?

To motivate our model, consider an airline company's sales process. For a flight on a given departure date, the airline company has its own sales channel and can observe its remaining inventory and dynamically adjust its price to influence the demand process. In addition, the company works with travel agents who interact with end customers and can thus use its incentive scheme to influence their sales efforts. As a one-off decision, the company's problem can be viewed as determining a booking limit that is higher than the capacity. However, the firm can keep track of bookings over time and continuously adjust its prices and commission rates for sales agents. For example, Qantas offers dynamic commissions that offer revenue opportunities for travel agents and induce dynamic effort exertion.[1] Because confirmed bookings may still be canceled prior to departure, airlines do not necessarily stop accepting bookings once they have reached their capacity. Rather, in such a case, the airline company has an opportunity to set a higher price or offer a lower-powered incentive scheme to the agents.

Another motivating example entails a deal website that relies on sales representatives to promote coupons for service providers. Generating excess demand is detrimental because, as Lee (2013, p. 238) points out, "A salon and spa service requires dedicated personal attention of the service provider. Therefore, excessive coupon sales create a scheduling nightmare and many customers cannot get timely service"; a leading cause of complaints on Yelp is scheduling problems caused by "over-capacity coupon sales". Yet, "commission-based sales representatives may be tempted to increase the number of coupons to sell beyond the capacity limit of the merchant". Thus, the firm has an opportunity to influence sales representatives' promotional efforts by dynamically adjusting the incentive scheme based on the remaining inventory.

Our analysis starts with the case of static pricing. We model the demand for the service as an arithmetic Brownian motion process with drift determined by the agent's unobserved effort level. The agent is effort averse and responds to a continuous-time incentive scheme provided by the firm. We characterize the solution to the firm's optimal incentive design problem and show the optimal incentive depends on both the remaining inventory and the time to service delivery. We prove that at any given time, the firm's value function is concave in the remaining inventory level. In other words, the firm faces a declining marginal value for each additional unit of inventory. Furthermore, the optimal sales incentive increases with the remaining inventory level, and the trajectory of the incentive can be rather volatile. We provide examples in which the optimal sales incentive surges toward the end of the time horizon.

[1] `https://bit.ly/qantasdc`.

Next, we consider a problem in which the firm can dynamically adjust both its incentive scheme and price. In this more general setting, the firm's demand follows an arithmetic Brownian motion process with drift determined by the price that the firm charges to end customers as well as the agent's unobserved effort level. The firm thus has two levers—its incentive scheme offered to the agent and its price—to influence the demand process. These two levers do not perfectly substitute for each other, because the agent's effort is unobservable, necessitating the firm's rent sharing with the agent, whereas the firm can unilaterally change its price at no cost (other than the change in the revenue rate). We observe that under dynamic pricing, the expected optimal incentive is relatively stable over the time horizon. Related to this observation, we show that under dynamic pricing, a static incentive scheme (i.e., a dynamic-pricing-only strategy) provides the firm with most of the benefits of a dynamic incentive scheme (i.e., a fully-dynamic strategy). Our finding is consistent with the observed practice in the airline industry that whereas firms routinely engage in dynamic pricing, the incentive schemes used for compensating sales agents are mostly static and do not typically vary over time (see, e.g., Alamdari 2002, Elmaghraby and Keskinocak 2003).

Realizing that a fully dynamic strategy can be challenging to implement in practice, we compare the performance of two partially dynamic strategies, under which either the pricing or contracting policy (but not both) is dynamic. Between these two partially dynamic strategies, the dynamic-pricing-only strategy outperforms the dynamic-contracting-only strategy under (1) a low initial inventory constraint and (2) a high price effect, and underperforms under (1) a high initial inventory constraint and (2) a low price effect. These findings provide guidance about how a partial dynamic strategy may fit into a firm, depending on the industry landscape it operates in.

Lastly, we demonstrate the value of the dynamic incentive scheme by re-examining the case of static pricing. We show that under static pricing, a static incentive scheme leads to a substantial efficiency loss compared with the corresponding dynamic incentive scheme. In other words, in the case in which service providers do not have price-setting power (e.g., due to market competition or regulatory constraints), a dynamic incentive structure offers a highly effective lever for the firm to boost and smooth customer demand.

To our best knowledge, our paper is the first in the literature to study a continuous-time incentive design problem with inventory and pricing considerations. In doing so, we develop a novel modeling framework that is applicable to problems facing the airline, hotel, and travel industries, among others. The rest of the paper is organized as follows. Section 2 briefly reviews the related literature. Sections 3 and 4 study incentive schemes under static and dynamic pricing, respectively. Section 5 compares the incentive schemes under static and dynamic pricing in the airline industry. Section 6 analyzes the case in which the firm operates on price segments. Finally, Section 7 concludes the paper. All proofs are in the appendix.

2. Literature

The vast majority of the moral hazard principal-agent theory literature, consistent with seminal papers such as those by Grossman and Hart (1983) and Holmstrom (1979), focuses on a *one-shot* interaction between a principal and an agent; see Laffont and Martimort (2002) for a comprehensive review of the literature. Departing from this tradition, in a seminal paper, Sannikov (2008) develops a continuous-time principal-agent framework that is applicable to a wide range of managerial settings. Building on the work by Sannikov (2008), the operations management and marketing literature has studied the incentive design problem for salesforce compensation (e.g., Rubel and Prasad 2016) and service contracting (e.g., Sun and Tian 2018). Our model builds on the framework developed by Sannikov (2008) (and in the ensuing literature) but has several distinguishing features. First, in our model, the sales outcome is constrained by a finite inventory level; if the materialized demand exceeds the inventory level, the firm may incur a cost of goodwill. Thus, unlike Sannikov (2008), we consider a situation in which the firm's profit is *non-monotone* in the demand generated by the agent. Second, we consider a case in which the firm can use dynamic pricing to influence demand, in addition to the dynamic compensation contract as studied by Sannikov (2008). Third, and perhaps most importantly, in contrast to the infinite-horizon established in the work by Sannikov (2008), we consider a *finite*-horizon continuous-time model with *limited* inventory, two key features facing many service industries.

Our paper also contributes to a stream of operations management literature that examines moral hazard and—more broadly—incentive design problems, including, for example, Alp and Şen (2021), Atasu, Ciocan, and Désir (2021), Baiman, Netessine, and Saouma (2010), Balachandran and Radhakrishnan (2005), Dai and Jerath (2013, 2016), Dai, Ke, and Ryan (2021), de Véricourt and Gromb (2018, 2019), Ke and Ryan (2018), Li, Chen, and Rong (2020), Long and Nasiry (2020), Nikoofal and Gümüş (2018), Plambeck and Zenios (2000, 2003), Serpa and Krishnan (2017), Song and Xiao (2021), and Yu and Kong (2020). Among these papers, the work by Plambeck and Zenios (2003) is a rare case with a continuous-time setup. They study a make-to-stock setting in which the principal employs an agent who chooses an unobservable production rate. The principal seeks to minimize the cost of holding inventory and backordering demand and uses an incentive contract that depends on the inventory level. Our paper differs from theirs in that we use a more standard continuous-time principal-agent framework that was developed by Sannikov (2008), and we consider a setting with perishable inventory that is prevalent in service industries. Dai and Jerath (2013, 2016) and Dai, Ke, and Ryan (2021) study the effect of demand censoring, which arises when demand in excess of inventory cannot be observed, on incentive design in a setting in which a principal and an agent interact exactly once. Different from this stream of literature that treats prices as exogenous and fixed, our paper builds on a rigorous continuous-time principal-agent framework that examines both incentive design and pricing decisions.

In the past few decades, an extensive literature on dynamic pricing under limited inventory has emerged; see Elmaghraby and Keskinocak (2003) for a review. Dynamic pricing has been a standard modeling feature in the large body of pricing and revenue management literature (e.g., Phillips 2005, Talluri and Ryzin 2005). We refer the reader to Chung, Ahn, and Chun (2021), den Boer and Keskin (2021), Harrison, Keskin, and Zeevi (2012), and Lei and Jasin (2020) for examples of recent advances and reviews of the literature. To our best knowledge, this literature does not consider the case in which the firm employs sales agents to influence demand. Accordingly, our paper incorporates an important consideration that this literature has overlooked—moral hazard in the sales-generation process. In doing so, our paper establishes a novel link between dynamic pricing and the moral hazard principal-agent theory.

Methodology-wise, our paper uses optimal control, which, despite its numerous applications in economics, marketing, and operations management (e.g., Sethi and Thompson 2000), remains uncommon in solving moral hazard problems, with an exception being the recent paper by Dai, Ke, and Ryan (2021). In addition, our paper is related to the economics and finance literature using stochastic calculus (e.g., Décamps et al. 2016, Keppo, Moscarini, and Smith 2008a). For example, as in Décamps et al. (2016), our monotonicity analysis derives from Ito's lemma, which is commonly used in mathematical finance. Our paper enriches this literature by connecting revenue management and moral hazard in a continuous-time setting.

3. Incentive Design under Static Pricing

A firm (i.e., the principal) hires a sales agent (i.e., the agent) to boost the demand for a service (e.g., air flight) offered at time T. At the beginning of the time horizon, that is, $t=0$, the principal has an inventory level I_0.[2] All the inventory will expire after time T. The firm cannot add new inventory at any time $t\in(0,T]$.

A standard Brownian motion $B=\{B_t,\mathcal{F}_t;0\le t\le T\}$ on $\{\Omega,\mathcal{F},\mathcal{Q}\}$ characterizes the uncertainties in the demand process. Specifically, D_t, the cumulative demand quantity for the product up to time $t\in[0,T]$, evolves according to

$$dD_t=(a+A_t)dt+\sigma dB_t, \tag{1}$$

where a corresponds to the firm's demand rate through its own sales channel, $A_t\ge 0$ is the agent's effort level, and σ is a constant diffusion term, which measures the volatility of the demand-generation process. The Brownian motion term (σdB_t) may lead to a negative drift in the demand

[2] The inventory level I_0 is exogenous in this model. The inventory I_0 can be extended to be endogenous and added as an optimization problem to the principal's objective function.

quantity, which corresponds to cancellations.[3] The agent's effort level $A = \{A_t, 0 \le t \le T\}$, which is a stochastic process by itself. The agent is effort-averse and incurs a cost of $c(A_t) = A_t^2/(2\eta)$, where the parameter η may be interpreted as the agent's effectiveness. The cost function implies the agent's cost of effort is convex and rising in A_t.

Several dynamic pricing and revenue management papers model demand quantity with a continuous-time Poisson process (see, e.g., Farias and Van Roy 2010). Bardina, Jolis, and Rovira (2000) show the Poisson process can be approximated by a stochastic process driven by a Brownian motion.[4] In this paper, we model the demand process directly with the arithmetic Brownian motion process in (1).[5]

At any time $t \in [0,T]$, the principal observes the cumulative demand up to time t (D_t) but not the agent's effort A_t. The sales up to time t is $S_t = \min\{D_t, I_0\}$, and the available inventory level is $I_0 - S_t$.[6] Because the agent's effort is unobservable to the principal, moral hazard arises.

Next, we consider the firm's and the agent's objective functions. We denote by ξ_T the compensation plan that the firm offers the agent. Stated differently, at $t = 0$, the firm commits to a take-it-or-leave-it compensation plan that leads to a terminal payoff of ξ_T that depends on the demand process D; the agent receives a payoff of ξ_T at the terminal time T if the agent accepts the compensation plan. Both the firm and the agent are risk neutral and share the same discount rate r; see Keppo, Touzi, and Zuo (2021) for the case in which the agent is risk averse. The agent chooses optimal effort $A = \{A_t, 0 \le t \le T\}$ by maximizing his expected utility:

$$V^{\mathrm{a}} = \sup_{\{A_t\}_{\{0\le t\le T\}}} E\left[e^{-rT}\xi_T - \int_0^T e^{-rt} c(A_t)ds\right], \tag{2}$$

and we write the agent's value function at any time $t \in [0,T]$ as

$$V_t^{\mathrm{a}} = \sup_{\{A_s\}_{\{t\le s\le T\}}} E\left[e^{-r(T-t)}\xi_T - \int_t^T e^{-r(s-t)} c(A_s)ds\right], \tag{3}$$

where ξ_T denotes the agent's terminal payoff, which is specified by some stochastic process dependent on the demand process D. The expectations in (2) and (3) are taken under the demand

[3] Our model abstracts from the strategic behavior of consumers whose cancellation decision may be driven by dynamic pricing. The focus of this paper is on the interaction between the principal and the agent, and potential strategic behavior of end consumers is beyond our scope.

[4] More generally, Brownian-motion-driven stochastic processes can approximate models with multiple small independent and identically distributed changes (see, e.g., Keppo, Moscarini, and Smith 2008b).

[5] Following the convention of the continuous-time moral hazard literature (e.g., Sannikov 2008), we use an arithmetic Brownian motion to capture demand uncertainty. Our main insights extend to the case of a geometric Brownian motion.

[6] In our model, inventory information is available to the firm but does not *directly* influence the agent's effort decision process. The firm dynamically adjusts the incentive it provides to the agent based on the remaining inventory, and the incentive directly drives the agent's effort decision process. Whether the agent has access to the inventory information does not affect our analysis.

dynamics (1). Given an initial condition $Y_0 \in \mathbb{R}$, we introduce the agent's payoff process Y^Z, which satisfies $\xi_T = Y_T^Z$:

$$Y_t^Z = Y_0 + \int_0^t \left(Z_s dD_s - H_{\mathrm{a}}(Z_s)ds + rY_s^Z ds\right) \text{ for all } t \in [0,T], \tag{4}$$

where Y_0 is a constant and its value is restricted to be greater than or equal to ρ_{a}, the agent's reservation utility (i.e., outside option). In (4), $H_{\mathrm{a}}(z)$ is the Hamiltonian function corresponding to the agent's value function (3):

$$H_{\mathrm{a}}(z) = \sup_{A_t} h(z, A_t), \tag{5}$$

where $h(z, A) = (a + A_t)z - c(A_t)$. Here, $(a + A_t)z$ is the instantaneous payment that the agent receives, $c(A_t)$ is the instantaneous cost of the agent's effort, and thus, $h(z, A)$ is the instantaneous profit of the agent. The agent maximizes the profit function $h(z, A)$ at each instant of time, which gives the Hamiltonian function in (5). We denote the optimizer of (5) as $\hat{A}_t(z)$. Because $c(A_t) = A_t^2/(2\eta)$, using the Hamiltonian maximizing condition (see, e.g., Sethi and Thompson 2000), we obtain the optimal effort and the corresponding Hamiltonian function:

$$\hat{A}_t(z) = \eta z \quad \text{and} \quad H_{\mathrm{a}} = \eta z^2/2 + za. \tag{6}$$

By (4), the incentive variable is denoted by Z_t, which can be viewed as the instantaneous incentive that the firm provides to the agent based on the cumulative demand up to time t (i.e., D_t). Thus, the incentive variable Z_t specifies the agent's value function changes with respect to the demand process D and the incentive process $\{Z_t\}$ is controlled by the firm: a higher Z_t provides a stronger incentive for the agent to exert sales effort, which boosts the expected demand.

The agent's terminal payoff $\xi_{\mathrm{T}} = Y_T^Z$ in (4) consists of a constant component Y_0 and an incentive component, which is linear in the demand process D. The constant term Y_0 is the agent's base pay. As long as the base pay is no less than the agent's outside option (i.e., $Y_0 \geq \rho_{\mathrm{a}}$), the agent accepts the contract and participates. The process Y tracks the agent's future utility, which is often referred to as the "promised utility" in the literature (see, e.g., Ljungqvist and Sargent 2004). Furthermore, as in the case of Sun and Tian (2018), Y_t can be viewed as the agent's performance score, which determines the agent's utility from continuing to work for the firm. In fact, by Cvitanić, Possamaï, and Touzi (2017), all contracts that satisfy the agent's participation constraint can be represented as $\xi_{\mathrm{T}} = Y_T^Z$ in (4). Thus, the payoff process (4) characterizes the agent's compensation in our model.

For ease of reference, we summarize in Table 1 the notation used in the paper.

Without loss of generality, we normalize the unit price to 1. The firm's problem is to choose the optimal agent's contract ξ_T to maximize its expected profit, as represented by

$$V^{\mathrm{p}} = \sup_{\xi_T} E\left[\int_0^T e^{-rt} dD_t - e^{-rT}\xi_T - \pi \cdot e^{-rT}\left(D_T - I_0\right)^+\right],$$

Table 1 Notation

A_t	The agent's effort level at time t
a	The demand rate through the firm's own sales channel
ξ_T	The agent's terminal payoff
D_t	Cumulative demand quantity at time t
S_t	Sales level up to time t
Z_t	Incentive variable at time t, denotes the agent's incentive pay to performance
T	Terminal time
I_0	Initial inventory level
p_t	Proportion of the price at time t
r	Discount rate
η	The agent's level of effectiveness
π	Overbooking penalty parameter
σ	Volatility of demand
α	Effect of price on the demand process
β	Effect of effort on the demand process
Γ	Static commission contract
b_0	Static commission fee

where the expectation is taken under the dynamics (1) with the agent's optimal effort and π is the overbooking penalty parameter and $\pi > 1$, meaning the overbooking penalty is greater than the unit price. The incentive compatibility (IC) constraint is the agent's optimization problem, and thus, the IC constraint is reflected in the demand dynamics (1). At a given time $t \in [0,T]$, the firm's expected profit is

$$V_t^{\mathrm{p}} = \sup_{\xi_T} E\left[\int_t^T e^{-r(s-t)} dD_s - e^{-r(T-t)}\xi_T - \pi e^{-r(T-t)}\left(D_T - I_0\right)^+\right]. \tag{7}$$

Next, we solve for the agent's optimal effort level and the firm's optimal compensation plan throughout the time horizon. The following proposition states that the agent's value function (3) coincides with the process (4) under contract ξ_T and shows the explicit form of the agent's optimal effort. The proof of this result is in the appendix.

PROPOSITION 1. *Under the contract representation $\xi_T = Y_T^Z$, the agent's value function $V_t^{\mathrm{a}} \le Y_t^Z$ for all $t \in [0,T]$, and the inequality is binding when the agent's effort is given by (6).*

Proposition 1 shows the agent's optimal effort can be derived using the Hamiltonian function (6). More specifically, given incentive Z_t at any time t, the agent's optimal effort is linear in the incentive variable Z_t; that is, $\hat{A}_t(Z_t) = \eta Z_t$.

LEMMA 1. *Given the payoff process characterized in (4) with terminal payoff $\xi_T = Y_T^Z$, the agent's value function evolves according to*

$$dV_t^{\mathrm{a}} = \left[-rV_t^{\mathrm{a}} + H_{\mathrm{a}}(Z_t)\right]dt + Z_t dD_t. \tag{8}$$

Lemma 1 sheds light on the agent's dynamic contract. The basic idea is that by (3), the agent's contract payoff depends on the value function, $V_T^{\mathrm{a}} = \xi_T$. As in Cvitanić, Possamaï, and Touzi (2017) and Keppo, Touzi, and Zuo (2021), we consider an agent's contract induced by the firm that solves a dynamic program. Our setup differs from Sannikov (2008)'s in that we consider a finite-time horizon and a limited capacity, whereas he considers an infinite-time horizon and imposes no capacity limits on the agent's output.

Given the agent's contract $\xi_T = Y_T^Z$, to motivate the agent to induce the desired effort level to boost the demand, the firm's problem is reduced to optimizing the process $\{Z_t\}$, for $0 \le t \le T$,

$$V^{\mathrm{p}} = \sup_{\{Z_t\}_{t\in[0,T]}, Y_0 \ge \rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt} dD_t - e^{-rT}\xi_T - \pi e^{-rT}\left(D_T - I_0\right)^+\right],$$

where the expectation is taken under the dynamics (1) with the agent's optimal effort, and ξ_T is given by (4).

Without loss of generality, we normalize both the agent's reservation utility ρ_{a} (i.e., outside option) and the demand effect of the firm's own sales a to zero, $\rho_{\mathrm{a}} = 0$ and $a = 0$. Then, the firm's problem is given by

$$V^{\mathrm{p}} = \sup_{\{Z_t\}_{t\in[0,T]}, Y_0 \ge 0} E\left[\int_0^T e^{-rt} dD_t - e^{-rT} Y_T^Z - \pi e^{-rT}\left(D_T - I_0\right)^+\right],$$

which, by (1), (4), and (6), can be written as

$$V^{\mathrm{p}} = \sup_{\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\eta Z_t dt - \int_0^T (e^{-rt}\eta Z_t^2/2) dt - \pi e^{-rT}\left(D_T - I_0\right)^+\right], \tag{9}$$

where the optimal Y_0 is equal to the normalized participation level ($\rho_{\mathrm{a}} = 0$), that is, $Y_0 = 0$. The firm's expected profit function at $t \in [0,T]$ is given by

$$V_t^{\mathrm{p}} = \sup_{\{Z_s\}_{s\in[t,T]}} E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s - \eta Z_s^2/2\right) ds - \pi e^{-r(T-t)}\left(D_T - I_0\right)^+\right].$$

Because V_t^{p} is a function of time t and demand D_t, by Ito's lemma, we obtain the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \sup_{Z_t}\left\{\left(\eta Z_t - \eta Z_t^2/2\right) - rV_t^{\mathrm{p}} + \frac{\partial V_t^{\mathrm{p}}}{\partial t} + \frac{\partial V_t^{\mathrm{p}}}{\partial D}\eta Z_t + \frac{1}{2}\frac{\partial^2 V_t^{\mathrm{p}}}{\partial D^2}\sigma^2\right\}. \tag{10}$$

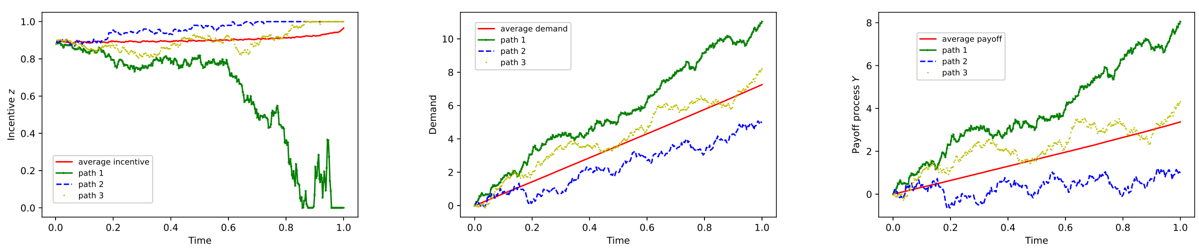
From (10) we get the following theorem, which specifies the optimal incentive variable.

Theorem 1. *The optimal incentive variable* Z_t *is given by* $\arg\sup_{Z_t}\left\{\eta Z_t - \frac{1}{2}\eta Z_t^2 + \frac{\partial V_t^{\mathrm{p}}}{\partial D}\eta Z_t\right\}$ *for all* $t \in [0,T]$.

We denote the firm's optimal incentive variable by Z_t^*. Note from (4) that Z_t^* provides the optimal compensation plan the firm offers to the agent.

3.1. Numerical Illustration

We use a numerical example to illustrate the firm's optimal incentive design problem (9) using a tree-grid method (Kossaczkỳ, Ehrhardt, and Günther 2019). Figure 1 shows how the optimal incentive variable Z_t and cumulative demand D_t evolve over time. Although the demand increases rather steadily over time, the trajectory of the optimal incentive variable can be volatile, especially when approaching the end of the time horizon. The dynamic contract is implemented as follows. The incentive variable Z_t is state dependent; that is, based on the realized demand quantity D_t at time t, the agent is provided with the incentive variable $Z_t(D_t)$ that can be interpreted as a piece rate at time t and influences the payoff process in (4) with terminal condition $\xi_{\mathrm{T}} = Y_T^Z$. Thus, the compensation ξ_T is the cumulative piece rate plus a fixed component, and the agent receives the compensation at time T. For instance, path 1 in Figure 1 shows the incentive variable Z_t changes dynamically as a function of the realized demand D_t. When the demand quantity of path 1 approaches the inventory constraint, the agent is provided with a low incentive Z_t and responds by lowering the sales effort.



(a) **Incentive versus time** (b) **Cumulative demand quantity versus time** (c) **Payoff process versus time**

Figure 1 **Incentive, demand quantity, and payoff processes with respect to time. Parameter values:** $r = 0.025$, $\eta = 8$, $I_0 = 10$, $\sigma = 2$, **and** $\pi = 2$.

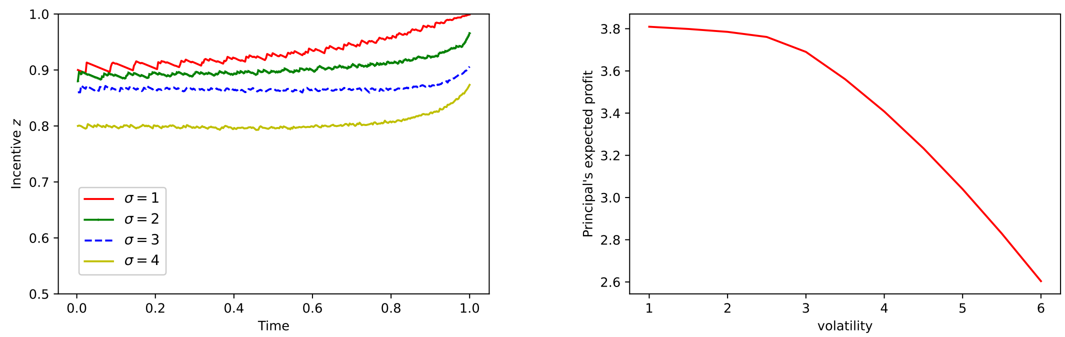
Next, we conduct comparative statics in terms of demand volatility σ, salesforce effectiveness η, and the discount rate r. Figure 2 shows the average incentive decreases in demand volatility σ. Two observations about Figures 3 and 4 may be made:

Observation 1 *The average effort first increases then decreases in the agent's effectiveness η.*

At the beginning of the time horizon, the firm faces ample inventory and desires a high sales effort level to help clear the inventory. Accordingly, it provides a more effective sales agent with a higher-power compensation plan, leading to higher cumulative demand and lower remaining inventory. Approaching the end of the time horizon, however, the firm is more concerned about overbooking; in view of the diminishing remaining inventory, it provides a more effective sales agent with a lower-power compensation plan (as shown in Figure 3(a)).

Observation 2 *In the early part of the time interval $[0,T]$, the average incentive increases in the discount rate r, and in the latter part of the time interval, the average incentive decreases in r.*

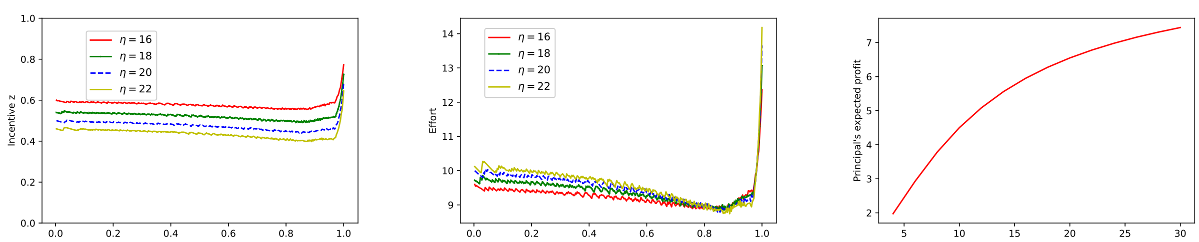
A large discount rate means the firm strongly prefers earlier sales to later sales. For this reason, it provides a higher-powered compensation plan to boost the demand for the service. Thus, the incentive first increases in the discount rate r. Because demand increases rapidly under high discount rates, the firm prefers slower cumulative demand growth toward the end of the horizon to avoid overbooking. Stated differently, as the service date approaches, the incentive decreases in the discount rate r (as shown in Figure 4(a)).



(a) **Incentive versus time**

(b) **The firm's expected profit versus demand volatility σ**

Figure 2 **Incentive and the firm's expected profit for different demand volatility σ. Parameter values: $r = 0.025$, $\eta = 8$, $I_0 = 10$, and $\pi = 2$.**

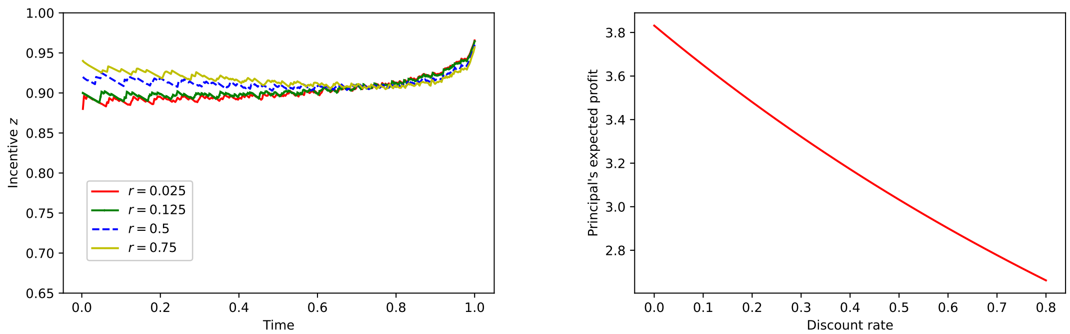


(a) **Incentive versus time**

(b) **Agent's effort versus time**

(c) **The firm's expected profit versus agent's effectiveness**

Figure 3 **Incentive, agent effort, and the firm's expected profit for different levels of salesforce effectiveness η. Parameter values : $r = 0.025$, $I_0 = 10$, $\sigma = 2$, and $\pi = 2$.**



(a) **Incentive versus time**

(b) **The firm's expected profit versus discount rate**

Figure 4 **Incentive sensitivity and the firm's expected profit for different discount rate** r**. Parameter values:** $\eta = 8$, $I_0 = 10$, $\sigma = 2$, **and** $\pi = 2$.

3.2. Structural Properties

We now present structural results that connect various market-environment variables to the firm's maximum profit (9) and optimal incentive scheme in Theorem 1.

First, we investigate the effect of the overbooking-penalty parameter and the initial inventory level on the firm's maximum profit.

PROPOSITION 2. *The firm's expected profit function* V_t^{P} *is (i) weakly decreasing in its overbooking-penalty parameter* π*, and (ii) weakly increasing in its initial inventory level* I_0*.*

As the overbooking penalty π increases, the firm is more concerned with situations in which the realized demand is greater than the supply (i.e., the inventory). Thus, all else being equal, the firm provides a lower-powered incentive scheme to its sales agent, which reduces the firm's expected sales and profit. As the initial inventory level I_0 increases, however, the firm has more supply to satisfy the demand and is less likely to overbook, all else being the same, which increases its expected profit.

The next proposition gives the concavity of the firm's profit with respect to the cumulative demand. We illustrate the proposition in Figure 5.

PROPOSITION 3. *For any time* $t \in [0, T]$*, the firm's expected profit function* V_t^{P} *is concave and decreasing in the cumulative demand* D_t*.*

As the cumulative demand D_t increases, the firm's remaining inventory level (i.e., $(I_0 - D_t)^+$) decreases. All else being the same, the firm's expected profit decreases. Consistent with this intuition, Proposition 3 states that the firm's expected profit decreases in the cumulative demand. On the flip side, the firm's expected profit increases in the remaining inventory. Meanwhile, Proposition 3 states that the firm faces a declining marginal value of an additional unit of inventory,

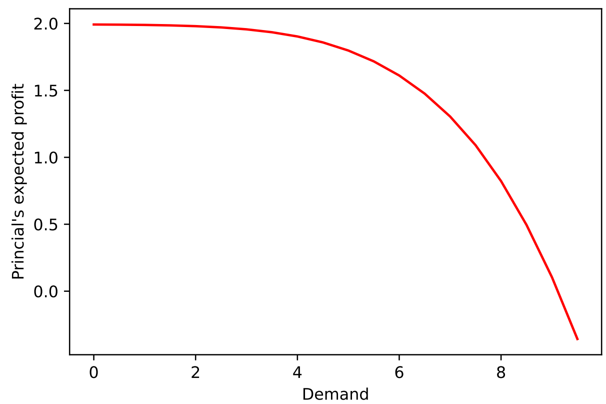


Figure 5 **The effect of cumulative demand on the firm's expected profit V_t^{p}. Parameter values:** $r = 0.025$, $\eta = 8$, $I_0 = 10$, $\sigma = 2$, $\pi = 2$, $T = 1$, **and** $t = 0.5$.

because translating a unit of inventory into sales becomes increasingly costly due to the moral hazard and the overbooking penalty.

The following proposition shows how the optimal incentive variable varies in the cumulative demand and the overbooking penalty.

PROPOSITION 4. *The optimal incentive variable Z_t^* decreases in the cumulative demand D_t and the overbooking penalty π.*

Proposition 4 states that the firm should reduce the incentive it offers to the agent when facing a lower inventory level (as a result of a higher cumulative demand) or a higher overbooking penalty. The single crossing property and monotone comparative statics (see, e.g., Milgrom and Shannon 1994) are the most commonly used methods in economics for comparative statics. Departing from these standard methods, our monotonicity analysis in Proposition 4 derives from the HJB equation in (10) and Ito's lemma, simlarly to Décamps et al. (2016). The basic idea underlying the proof of Proposition 4 (see the appendix) is to represent the firm's expected profit function V_t^{p} as a function of parameter π and then take the first-order derivative with respect to π. To do so, we substitute the optimal incentive variable Z_t given by Theorem 1 into the HJB equation (10) and use Ito's lemma to establish Proposition 4.

4. Incentive Design under Dynamic Pricing

So far, we have assumed the price is exogenously given and focused on the case in which the demand process follows an arithmetic Brownian process, with drift solely dependent on the agent's unobservable effort. We now extend the model to allow the drift to be dependent on both the agent's effort and the firm's selling price. In this case, the firm's problem consists of (i) designing the incentive for the agent and (*ii*) selecting a dynamic pricing policy for the product (or the service).

We start with the demand model. Let p_t denote the price of the product at time t, and thus, $\{p_t\}_{t\in[0,T]}$ is the price process. Without loss of generality, p_t is scaled and restricted to be in the range $(0,1]$ at any time $t\in[0,T]$. The cumulative demand quantity follows

$$dD_t = [\alpha(1-p_t)+\beta A_t]\,dt+\sigma dB_t, \tag{11}$$

where $\sigma>0$ is the volatility of the demand, and $\alpha\geq 0$ and $\beta\geq 0$ capture the effects of price p_t and agent's effort A_t on the demand quantity, respectively. Related to (1), the term $\alpha(1-p_t)$ in the drift term captures the price effect for both the firm's own sales channel and the agent. The Brownian motion term (σdB_t) may lead to negative changes in the cumulative demand, which corresponds to cancellations.

Given the agent's compensation process defined in (4), the agent's terminal payoff satisfies $\xi_T = Y_T^Z$. Corresponding to the demand process (11) and the agent's value function (3), the Hamiltonian function is as follows:

$$H_{\mathrm{a}}(z) = \sup_{A_t} h(z,A_t), \tag{12}$$

where $h(z,A) = z\,[\alpha(1-p_t)+\beta A_t]-c(A_t)$. Similar to (5), the agent maximizes the profit function $h(z,A)$ at each instant of time, which gives the Hamiltonian function in (12). We obtain the optimal agent's effort

$$\hat{A}_t(z)=\eta\beta z \quad \text{and} \quad H_{\mathrm{a}} = \eta\beta^2 z^2/2+z\alpha(1-p_t). \tag{13}$$

By (6), the payoff process Y^Z in (4) can be reformulated as

$$Y_t^Z = Y_0+\int_0^t \left(\eta\beta^2 Z_s^2/2+rY_s^Z\right)ds+Z_s\sigma dB_s \text{ for all } t\in[0,T]. \tag{14}$$

The firm's value function is given by

$$\begin{aligned} V^{\mathrm{p}} &= \sup_{\{p_t\}_{t\in[0,T]},\xi_T} E\left[\int_0^T e^{-rt}p_t dD_t - e^{-rT}\xi_T - \pi e^{-rT}\left(D_T-I_0\right)^+\right] \\ &= \sup_{\{p_t\}_{t\in[0,T]},\{Z_t\}_{t\in[0,T]},Y_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt}p_t dD_t - e^{-rT}Y_T^Z - \pi e^{-rT}\left(D_T-I_0\right)^+\right], \end{aligned} \tag{15}$$

which, after substituting D_t and Y_T^Z with (11) and (14), respectively, can be rewritten as

$$\begin{aligned} V^{\mathrm{p}} = \sup_{\{p_t\}_{t\in[0,T]},\{Z_t\}_{t\in[0,T]}} E\Bigg[&\int_0^T e^{-rt}(p_t\eta\beta^2 Z_t+\alpha p_t-\alpha p_t^2)dt-\left(\rho_{\mathrm{a}}+\int_0^T e^{-rt}\eta\beta^2 Z_t^2/2dt\right) \\ &-\pi e^{-rT}\left(D_T-I_0\right)^+\Bigg], \end{aligned} \tag{16}$$

where the optimal Y_0 is equal to the normalized participation level ($\rho_{\rm a}=0$); that is, $Y_0=0$. The expectation is taken under the dynamics (11) with the agent's optimal effort $\hat{A}_t(Z_t)$ in (13). The IC constraint is the agent's optimization problem, and thus, the IC constraint is reflected in the demand dynamics (11). The firm's expected profit function at $t\in[0,T]$ is given by

$$\begin{aligned}
V_t^{\rm p} &= \sup_{\{p_s\}_{s\in[t,T]},\{Z_s\}_{s\in[t,T]}} E_t\left[\int_t^T e^{-r(s-t)}\left(p_s\eta\beta^2Z_s+\alpha p_s-\alpha p_s^2-\eta\beta^2Z_s^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\
&=\sup_{p_t,Z_t}\left(p_t\eta\beta^2Z_t+\alpha p_t-\alpha p_t^2-\eta\beta^2Z_t^2/2\right)\Delta t\\
&\quad+(1-r\Delta t)\left[V_t^{\rm p}+\frac{\partial V_t^{\rm p}}{\partial t}\Delta t+\frac{\partial V_t^{\rm p}}{\partial D}(\eta\beta^2Z_t+\alpha-\alpha p_t)\Delta t+\frac{1}{2}\frac{\partial^2V_t^{\rm p}}{\partial D^2}\sigma^2\Delta t\right]+O\left((\Delta t)^2\right),
\end{aligned}\tag{17}$$

where Δt is a small time change and $O(\cdot)$ represents the rate at which the function approximates the actual value. Then, we have the following HJB equation:

$$\sup_{p_t,Z_t}\left\{\left(p_t\eta\beta^2Z_t+\alpha p_t-\alpha p_t^2-\eta\beta^2Z_t^2/2\right)-rV_t^{\rm p}+\frac{\partial V_t^{\rm p}}{\partial t}+\frac{\partial V_t^{\rm p}}{\partial D}(\eta\beta^2Z_t+\alpha-\alpha p_t)+\frac{1}{2}\frac{\partial^2V_t^{\rm p}}{\partial D^2}\sigma^2\right\}=0,\tag{18}$$

which gives the following theorem.

Theorem 2. *The firm's optimal incentive variable Z_t and optimal price p_t are given by*

$$\arg\sup_{Z_t,p_t}\left\{\left(p_t\eta\beta^2Z_t+\alpha p_t-\alpha p_t^2-\eta\beta^2Z_t^2/2\right)+\frac{\partial V_t^{\rm p}}{\partial D}\left(\eta\beta^2Z_t-\alpha p_t\right)\right\}\ \textit{for all}\ t\in[0,T].\tag{19}$$

Theorem 2 leads to several structural properties. First, we obtain the following result.

Proposition 5. *The firm's expected profit function $V_t^{\rm p}$ decreases in the overbooking penalty π.*

To derive structural properties of the firm's optimal price and compensation plan, we make the following assumption in the rest of this section.

Assumption 1. *The price effect is bounded below; more specifically, $\alpha>\eta\beta^2$.*

The above condition indicates that price changes have a sufficiently large effect on demand; that is, customer demand is sufficiently elastic. (Later, in Section 5, we show the model parameters calibrated from the airline industry satisfy this condition.)

Next, we examine the concavity of the firm's expected profit function with respect to cumulative demand D_t.

Proposition 6. *The firm's expected profit function $V_t^{\rm p}$ is concave and decreasing in cumulative demand D_t.*

As in Figure 5, a lower cumulative demand level (D_t) means a higher remaining inventory level $(I_0-D_t)^+$; thus, the firm faces a higher future sales potential. Furthermore, the marginal value of the inventory level decreases, which gives rise to the concavity property.

The following proposition describes how the optimal incentive variable changes with respect to the cumulative demand and the overbooking penalty.

PROPOSITION 7. *Under a dynamic pricing and dynamic contracting strategy,*

(i) the optimal incentive variable Z_t^* decreases in both the cumulative demand D_t and the unit overbooking penalty π;

(ii) the optimal price p_t^* increases in both the cumulative demand D_t and the unit overbooking penalty π.

As the cumulative demand D_t increases, less inventory becomes available. Then, the firm has a lower incentive to boost its demand. Accordingly, it does not lower the price or provide a higher incentive to the agent. As the overbooking penalty parameter π increases, the firm is motivated to avoid overbooking by inhibiting demand growth. Therefore, the optimal incentive variable Z_t^* decreases and the optimal price p_t^* increases in the unit overbooking penalty π.

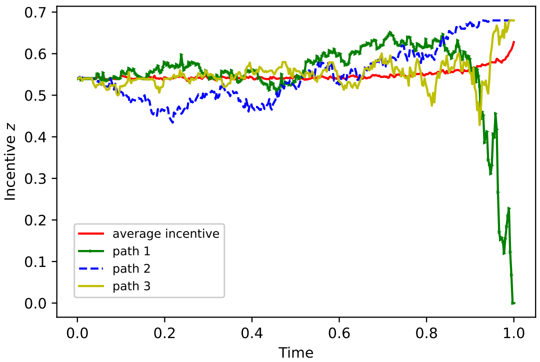
4.1. Does a Static Incentive Contract Suffice?

We now illustrate the firm's optimal pricing and incentive decisions with a numerical example in Figure 6. In the view of Figure 6, the firm dynamically adjusts its incentive scheme and selling price to manage its revenue while avoiding overbooking. Path 1 in Figure 6, for example, corresponds to the case in which the optimal price is greater than the baseline price (normalized to one). Whereas higher prices reduce the likelihood of overbooking, they also inhibit demand growth; thus, the optimal price is not always greater than the baseline price.

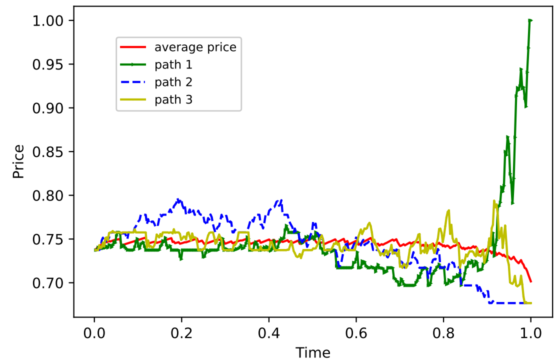
We observe from Figure 6 that under dynamic pricing, the expected optimal incentive is relatively stable. This observation brings up the question of how a partially dynamic strategy performs in comparison to the fully dynamic strategy. (Note airline companies have used a partially dynamic strategy, because they are known to use dynamic pricing but do not frequently adjust their incentive schemes for sales agents.) As such, we examine whether a static incentive contract is sufficient when the firm adjusts its price dynamically.

Consider the case in which the firm offers the agent a static commission contract in the form of $\Gamma = a_0 + \int_0^T e^{-r(t-T)} b_0 dD_t$; by substituting ξ_T in (3) with the static commission contract Γ, the agent's optimal response to this static contract is given as the optimizer to the agent's value function in (3), i.e., the optimal effort is given by $\hat{a} = \eta\beta b_0$. We can numerically solve the optimal incentive scheme and pricing policy under this static contract. (Note a_0 can be determined by the agent's reservation utility.) As Figure 7(a) illustrates, the performance gap between a fully dynamic strategy (dynamic contracting with dynamic pricing) and a partially dynamic strategy (static contracting with dynamic pricing) is below 13%, meaning that using dynamic pricing alone can achieve most of the benefits of a fully dynamic strategy.

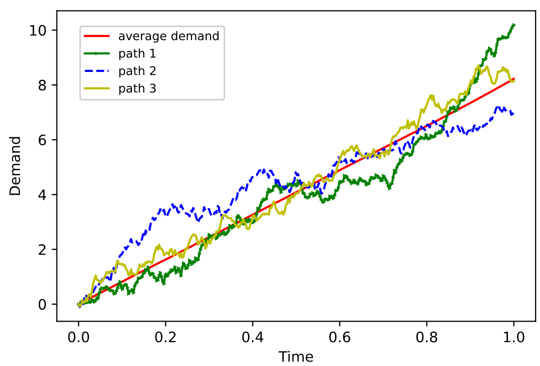
This observation has important implications because implementing a fully dynamic strategy is challenging in certain cases. In the airline industry, for example, whereas firms commonly adjust



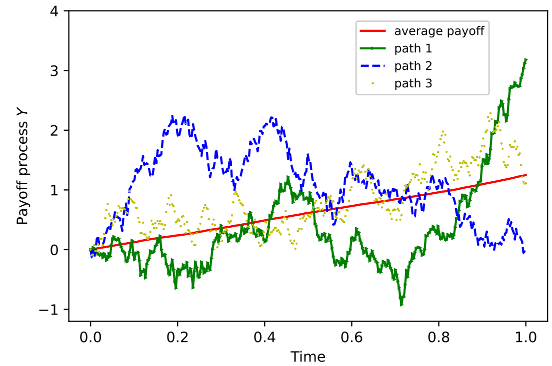
(a) **Incentive versus time**



(b) **Price versus time**



(c) **Demand versus time**



(d) **Payoff process versus time**

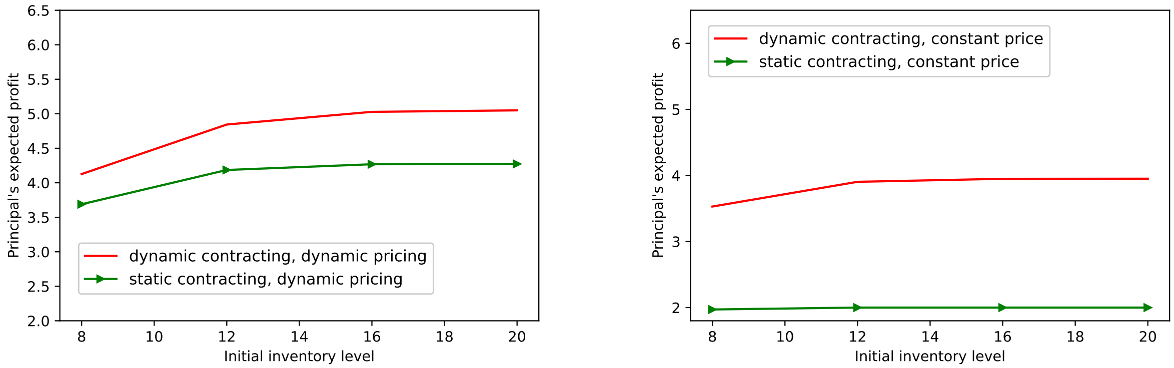
Figure 6 **Incentive, price, demand quantity and payoff process with respect to time. Parameter values:** $r = 0.025$, $\eta = 8$, $I_0 = 10$, $\sigma = 2$, $\alpha = 15$, $\beta = 1$, **and** $\pi = 2$.

their fare prices rather frequently, implementing a change in the way they compensate sales agents is not a straightforward process (see, e.g., Alamdari 2002, Elmaghraby and Keskinocak 2003). This observation means that when dynamic pricing is possible, the firm can still manage to extract most of the benefits from a fully dynamic strategy.

To be certain, dynamic contracting can be advantageous when a firm lacks control over pricing (e.g., due to market competition or regulatory reasons). The example in Figure 7(b) demonstrates the value of dynamic contracting in the presence of static pricing. In this case, we show relying on a static (albeit optimized) incentive scheme can result in a significant loss of efficiency.

5. Comparison across Strategies

When a fully-dynamic strategy is hard to implement, a firm may consider partially dynamic strategies, in which either the pricing or contracting policy (but not both) is dynamic. In this section, we compare four different strategies: (i) fully-dynamic (i.e., dynamic contracting and dynamic pricing), (ii) dynamic-pricing-only (i.e., static contracting and dynamic pricing), (iii) dynamic-contracting-only (i.e., dynamic contracting and static pricing), and (iv) fully-static (i.e., static contracting and



(a) **The firm's expected profit versus the initial inventory level**

(b) **The firm's expected profit versus the initial inventory level**

Figure 7 **The firm's expected profit with respect to initial inventory level under different contracting and pricing strategies. Parameter values:** $r = 0.025$, $\eta = 8$, $I_0 = 10$, $\sigma = 2$, $\alpha = 15$, $\beta = 1$, **and** $\pi = 2$.

static pricing). As discussed in the previous section, the dynamic-pricing-only strategy has been used in the airline industry, in which dynamic pricing—in the absence of dynamic contracting—is the norm. Our comparison across strategies in this section sheds light on how the performance of different strategies depends on the operating environment.

We denote by $V_{\mathrm{p}}^{\mathrm{fd}}$, $V_{\mathrm{p}}^{\mathrm{dp}}$, $V_{\mathrm{p}}^{\mathrm{dc}}$, and $V_{\mathrm{p}}^{\mathrm{fs}}$ the firm's expected profit under the fully-dynamic, dynamic-pricing-only, dynamic-contracting-only, and fully-static strategies, respectively. Note that the firm's value function under a fully-dynamic strategy, $V_{\mathrm{p}}^{\mathrm{fd}}$, is the same as the firm's value function in (17) in Section 4. More specifically, the value functions are given as follows:

$$V_{\mathrm{p}}^{\mathrm{fd}} = \sup_{\{p_t\}_{t\in[0,T]},\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p_t\eta\beta^2 Z_t + \alpha p_t - \alpha p_t^2 - \eta\beta^2 Z_t^2/2\right)dt - \pi e^{-rT}\left(D_T - I_0\right)^+\right] \tag{20}$$

$$V_{\mathrm{p}}^{\mathrm{dp}} = \sup_{\{p_t\}_{t\in[0,T]},b_0} E\left[\int_0^T e^{-rt}\left(p_t\eta\beta^2 b_0 + \alpha p_t - \alpha p_t^2 - \eta\beta^2 b_0^2 dt\right) - \pi e^{-rT}\left(D_T - I_0\right)^+\right] \tag{21}$$

$$V_{\mathrm{p}}^{\mathrm{dc}} = \sup_{p,\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 Z_t + \alpha p - \alpha p^2 - \eta\beta^2 Z_t^2/2dt\right) - \pi e^{-rT}\left(D_T - I_0\right)^+\right] \tag{22}$$

$$V_{\mathrm{p}}^{\mathrm{fs}} = \sup_{p,b_0} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 b_0 + \alpha p - \alpha p^2 - \eta\beta^2 b_0^2 dt\right) - \pi e^{-rT}\left(D_T - I_0\right)^+\right]. \tag{23}$$

More details about how we obtained (20)–(23) can be found in the proof of Proposition 8 in the appendix. Clearly, the fully dynamic strategy dominates the others; that is, $V_{\mathrm{p}}^{\mathrm{fd}} \geq V_{\mathrm{p}}^{\mathrm{dp}}$, $V_{\mathrm{p}}^{\mathrm{fd}} \geq V_{\mathrm{p}}^{\mathrm{dc}}$, and also naturally $V_{\mathrm{p}}^{\mathrm{dp}} \geq V_{\mathrm{p}}^{\mathrm{fs}}$ and $V_{\mathrm{p}}^{\mathrm{dc}} \geq V_{\mathrm{p}}^{\mathrm{fs}}$. Note when the firm searches over all admissible fully dynamic strategies, it also considers all admissible partially and fully static strategies. Comparing $V_{\mathrm{p}}^{\mathrm{dp}}$ and $V_{\mathrm{p}}^{\mathrm{dc}}$, on the other hand, is not straightforward and depends on the interaction of pricing and effort parameters.

The following proposition illustrates the effectiveness of the dynamic contract, which allows the firm to adjust the sales incentive in response to the realized demand and the remaining inventory. For example, when the current cumulative demand level D_t is high, the firm might reduce the incentive (or bonus) for the sales agent to avoid overbooking to match the initial inventory. Compared with the case of a static contract, using a dynamic contract incurs a lower instantaneous cost of incentivizing the agent. More specifically, we compare the value functions $V_{\mathrm{p}}^{\mathrm{fd}}$ (under the fully-dynamic strategy) in (20) and $V_{\mathrm{p}}^{\mathrm{dp}}$ (under the dynamic-pricing-only strategy) in (21). For each unit of bonus provided, the instantaneous incentive cost of the dynamic contract $\eta\beta^2/2$ is half of the static contract's cost $\eta\beta^2$.

PROPOSITION 8. *In view of (20) to (23), the static contract (i.e., $\Gamma = a_0 + \int_0^T e^{-r(t-T)} b_0 dD_t$) induces an instantaneous incentive cost of $\eta\beta^2 b_0^2$ to the principal, whereas the dynamic contract defined in (4) induces an instantaneous incentive cost of $\eta\beta^2 Z_t^2/2$.*

In our ensuing numerical study, we derive our baseline parameter values using simulated data from the airline industry. We calibrate the parameters for the airline industry based on Perera and Tan (2019) using the following procedure. We first define the parameter space for our model. For each pair of parameters, we derive the optimal dynamic-pricing-only policy using the tree-grid method (Kossaczkỳ, Ehrhardt, and Günther 2019). Then, for each pair of parameters, we simulate the demand data based on our Brownian motion driven demand process model in (11). Finally, we compare and choose the parameters that give the smallest distance between the simulated demand data based on Perera and Tan (2019) and the demand data based on our model.[7]

Next, we compare the four strategies—as defined in (20) to (23)—under various overbooking penalty parameters and illustrate the results in Figure 8. Note from the figure that the profit difference between the fully dynamic strategy and the dynamic-pricing-only strategy is around 1%. In other words, under the parameters drawn from the airline industry, a firm can achieve approximately 99% of the maximum expected profit by practicing a dynamic-pricing-only strategy, and thus, offering a static incentive scheme to sales agents.[8] In addition, under the estimated parameters, the dynamic-pricing-only strategy dominates the dynamic-contracting-only strategy. The reason is that under the estimated parameters, the effect of pricing on demand is greater than that of the sales incentives. As such, dynamic pricing is more effective than dynamic contracting.

[7] The estimated benchmark parameter values are $r = 0.025$, $\eta = 10$, $I_0 = 150$, $\sigma = 13$, $\alpha = 354$, $\eta\beta^2 = 10$, and $\pi = 2$. Because $\eta\beta^2$ represents the effort effect on the demand, without loss of generality, we let $\beta = 1$ and $\eta = 10$. Note the estimated model parameters satisfy Assumption 1. In addition to these benchmark parameter values, we have also tested cases where α is not far greater than β, and show that our main results extend qualitatively to a wide range of parameter values.

[8] The optimality gap is around 13% in Figure 7a, because demand is more sensitive to pricing in that scenario.

More specifically, the dynamic-pricing-only strategy is more effective in smoothing the uncertain demand and avoiding overbooking. Our results indicate the dynamic-pricing-only strategy performs nearly as well as the fully dynamic strategy, consistent with the prevailing practice in the airline industry, which places heavy emphasis on dynamic pricing but largely relies on static incentive schemes for sales agents (see, e.g., Alamdari 2002, Elmaghraby and Keskinocak 2003).

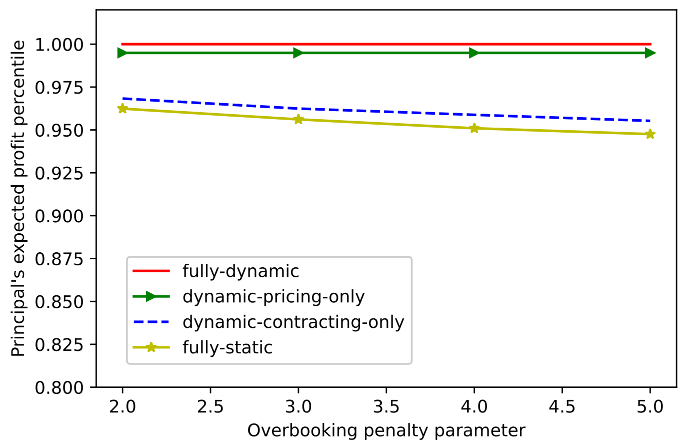


Figure 8 **The firm's expected profit versus overbooking penalty parameter π. The parameter values, as calibrated from Perera and Tan (2019), are $r = 0.025$, $\eta = 10$, $I_0 = 150$, $\sigma = 13$, $\alpha = 354$, and $\beta = 1$.**

Next, Figures 9 to 11 illustrate how different combinations of model parameters influence the firm's expected profits from different strategies.

A salient feature of our model is limited inventory. Thus, investigating the effect of limited inventory on the performance of various strategies is of interest. Figure 9 shows that as the inventory constraint becomes tighter (i.e., as I_0 is decreases), the performance differences between various strategies widen. One factor accounting for the performance difference is that, all else being equal, a lower initial inventory leads to a higher probability of overbooking. For this reason, the firm faces a tradeoff between boosting demand and avoiding overbooking penalty term $\pi e^{-rT}\left(D_T - I_0\right)^+$. Thus, a dynamic strategy helps achieve substantially better performance than a static one. Because the price effect on demand is greater than the effort effect, the dynamic-pricing-only strategy performs better than the dynamic-contracting-only strategy under the estimated parameters for the airline industry.

Another salient feature of our model is demand uncertainty, which is closely relevant to moral hazard: in the absence of demand uncertainty, the firm can establish a one-to-one correspondence between outcome and effort, so moral hazard does not arise. Figure 10 illustrates how

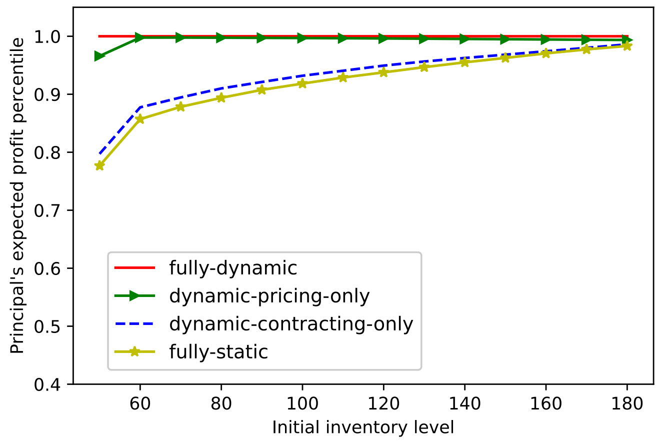


Figure 9 **The firm's expected profit percentile versus the initial inventory level (I_0). All the parameter values are the same as in Figure 8 except that we vary I_0 between 50 and 180.**

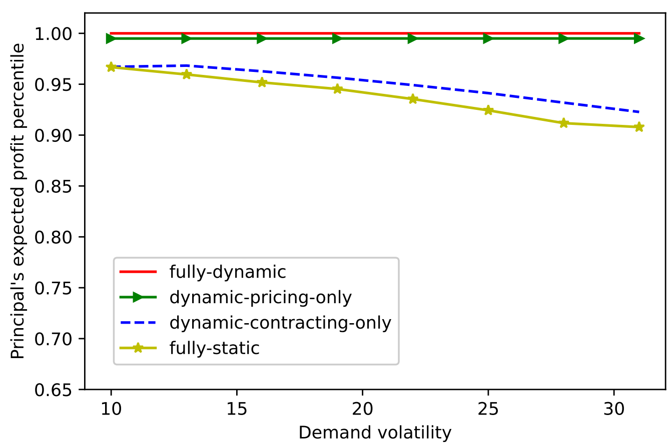


Figure 10 **The firm's expected profit percentile versus demand volatility (σ). All parameter values are the same as in Figure 8 except that we vary σ between 10 and 30.**

demand volatility influences different strategies. As demand volatility rises, the performance differences between the fully dynamic strategy, the two partially dynamic strategies (esp. the dynamic-contracting-only strategy), and the fully static strategy widen. In addition, Figure 10 shows that when demand is sufficiently volatile, the dynamic-pricing-only strategy dominates the dynamic-contracting-only strategy. The reason is that when the price-effect parameter α is relatively high (as in the case of the airline industry), dynamic pricing is more effective in smoothing the demand process than dynamic contracting.

Next, as Section 4 suggests, the relative magnitude of the price and effort effects plays an instrumental role in influencing the performance differences across strategies. Figure 11 shows that

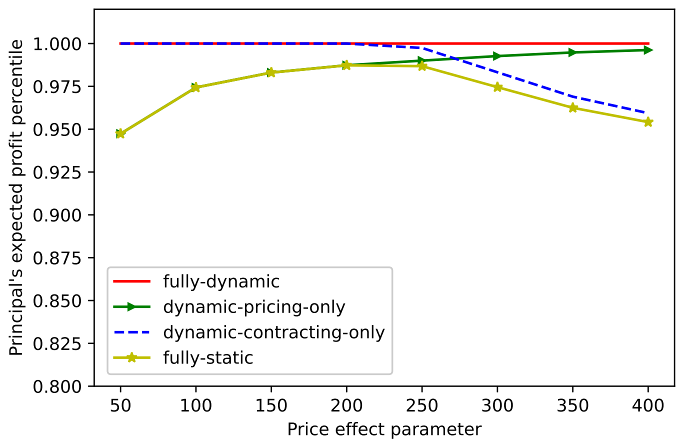
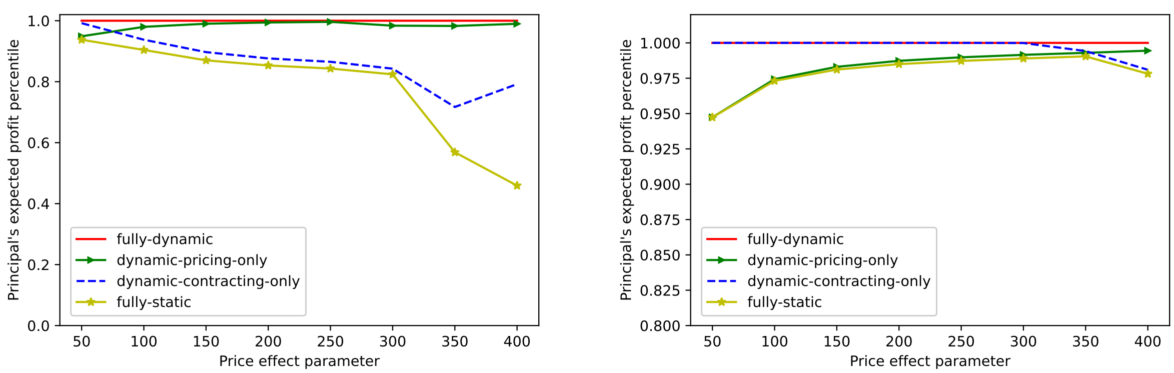


Figure 11 **The firm's expected profit percentile versus price-effect parameter (α). All parameter values are the same as in Figure 8 except that we vary α between 50 and 400. In particular, the initial inventory level $I_0 = 150$.**



(a) $I_0 = 50$ (b) $I_0 = 200$

Figure 12 **The firm's expected profit percentile versus price-effect parameter (α) under different initial inventory levels. Both of the above panels use the same parameter values as in Figure 11 except that we use different values of I_0.**

when the price effect α is low, dynamic contracting with static pricing dominates the strategy of static contracting with dynamic pricing; that is, $V_{\mathrm{p}}^{\mathrm{dc}} \geq V_{\mathrm{p}}^{\mathrm{dp}}$. Note that when the price-effect parameter is low, pricing has little effect on demand. As a result, the firm prefers to influence demand over the planning horizon through dynamic incentive design.

Furthermore, Figure 12 provides the impact of the price-effect parameter (α) on the comparison of the strategies under different initial inventory levels (I_0). By comparing the crossing points at which the dynamic-pricing-only and dynamic-contracting-only strategies achieve the same performance in Figure 11 (in which $I_0 = 150$) and the left panel of Figure 12 (in which $I_0 = 50$), we observe that under a low initial inventory level, the dynamic-pricing-only strategy outperforms the dynamic-contracting-only strategy within a wider range of price-effect parameters. Likewise, by comparing the crossing points in Figure 11 (in which $I_0 = 150$) and the right panel of Figure 12 (in which $I_0 = 200$), we observe that under a high inventory level, the dynamic-contracting-only strategy tends to be more effective than the dynamic-pricing-only strategy.

One may wonder whether pricing and contracting strategies are substitutes or complements. We observe from Figures 11 and 12 that the pricing and contracting strategies are mostly substitutes, because in most cases either the dynamic-pricing-only or the dynamic-contracting-only strategy helps the firm achieve most of the benefits of the fully dynamic strategy. Note that the pricing strategy follows from revenue management, where, for example, a price increase will increase revenue if demand is inelastic. On the other hand, a higher incentive for the sales agent incurs costs, but it may increase sales at a high price.

Finally, we conduct convergence analysis to shed light on the relative performance of the dynamic-pricing-only and dynamic-contracting-only strategies. The following corollary provides the comparison as several basic parameters approach zero or infinity.

Proposition 9. (*i*) *As the demand sensitivity in terms of effort, denoted by β, approaches zero or infinity, the dynamic-pricing-only strategy outperforms the dynamic-contracting-only strategy.*

(*ii*) *As the demand sensitivity in terms of pricing, denoted by α, approaches zero or infinity, the dynamic-contracting-only strategy outperforms the dynamic-pricing-only strategy.*

(*iii*) *As the initial inventory I_0 approaches infinity, the dynamic-contracting-only and dynamic-pricing-only strategies become equivalent to each other.*

(*iv*) *As the volatility σ approaches zero or infinity, the dynamic-contracting-only and dynamic-pricing-only strategies become equivalent to each other.*

6. Analysis of Price Segmentation

We have introduced a general modeling framework that can be built upon to incorporate various realistic features. Motivated by the airline industry (see, e.g., Feng and Xiao 2001), in this section, we incorporate the use of price segments that are released gradually and consecutively, and generally the price segments released as time approaches are more expensive.

6.1. Fixed Switching Point

We start by considering the case in which the switching point between the two price levels (i.e., p_1 and p_2) is exogenous. Without loss of generality, we consider the case in which the switching point is in the middle of the planning horizon, such that the selling price is p_1 for the first part of the planning horizon and p_2 for the rest. The cumulative demand quantity follows (11). Further, as in (11), the term $\alpha(1-p_t)$ in the drift term captures the price effect for both the firm's own sales channel and the agent. As in Section 4, the agent's optimal effort is given by (13).

However, now the firm's value function is given by

$$V^{\mathrm{p}} = \sup_{\{p_1,p_2\}\{z_t\}_{t\in[0,T]}} E\left[\int_0^{T/2} p_1 e^{-rt} dD_t + \int_{T/2}^{T} p_2 e^{-rt} dD_t - e^{-rT}\xi_T - \pi e^{-rT}(D_T - I_0)^+\right]. \tag{24}$$

Then, for fixed (p_1,p_2), we have the following HJB equation:

$$\sup_{Z_t}\left\{\left(p_t\eta\beta^2 Z_t + \alpha p_t - \alpha p_t^2 - \eta\beta^2 Z_t^2/2\right) - rV_t^{\mathrm{p}} + \frac{\partial V_t^{\mathrm{p}}}{\partial t} + \frac{\partial V_t^{\mathrm{p}}}{\partial D}(\eta\beta^2 Z_t + \alpha - \alpha p_t) + \frac{1}{2}\frac{\partial^2 V_t^{\mathrm{p}}}{\partial D^2}\sigma^2\right\} = 0, \tag{25}$$

where $p_t = p_1$ if $t \le T/2$ and $p_t = p_2$ if $t \ge T/2$. This gives the firm's optimal incentive variable Z_t, which solves

$$\arg\sup_{Z_t}\left\{\left(p_t\eta\beta^2 Z_t - \eta\beta^2 Z_t^2/2\right) + \frac{\partial V_t^{\mathrm{p}}}{\partial D}\eta\beta^2 Z_t\right\} \text{ for all } t\in[0,T]. \tag{26}$$

Then, we can search for the pair of (p_1,p_2) that maximizes the firm's value.

Here, we present numerical results for the price-segment problem with dynamic contracting. In Figure 13, we fix the first price segment and compute the principal's value with respect to different second price segments. The first price in the green line is optimal under these parameters. We observe that the optimal second price is lower than the optimal first price in the green line. However, the opposite is true for the blue line, which is dominated by the pricing strategy in the green line. The relationship between the two price segments is not straightforward: In the first segment, the principal aims to generate more revenue by setting a higher price. Simultaneously, a higher dynamic incentive is provided, which is more efficient in boosting demand. In the second segment, the principal aims to slow down the increase in demand and avoid overbooking by setting a low incentive and low price.

Next, let us consider the price-segment problem with static contracting, that is, $\Gamma = a_0 + \int_0^T e^{-r(t-T)} b_0 dD_t$. The firm's value function is given by

$$\begin{aligned} V^{\mathrm{p}} &= \sup_{\{p_1,p_2,\Gamma\}} E\left[\int_0^{T/2} p_1 e^{-rt} dD_t + \int_{T/2}^{T} p_2 e^{-rt} dD_t - e^{-rT}\Gamma - \pi e^{-rT}(D_T - I_0)^+\right] \\ &= \sup_{p_1,p_2,b_0} E\left[\int_0^{T/2} e^{-rt}\left(p_1\eta\beta^2 b_0 + \alpha p_1 - \alpha p_1^2 - \eta\beta^2 b_0^2\right) dt\right. \\ &\quad \left. + \int_{T/2}^{T} e^{-rt}\left(p_2\eta\beta^2 b_0 + \alpha p_2 - \alpha p_2^2 - \eta\beta^2 b_0^2\right) dt - \pi e^{-rT}\left(D_T - I_0\right)^+\right]. \end{aligned} \tag{27}$$

We can analytically solve for the optimal p_1, p_2, and b_0, and we find $p_1 \le p_2$. Without discounting $(r=0)$, p_1 and p_2 are interchangeable in the objective function, and we have $p_1 = p_2$. Otherwise, the following lemma shows the optimal first price is weakly lower than the second price.

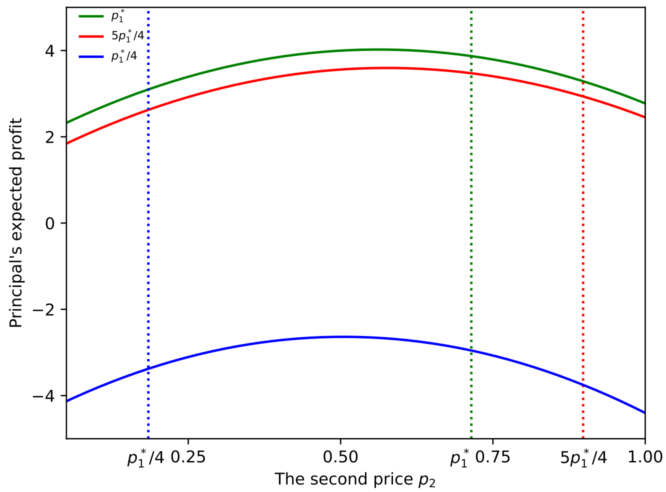


Figure 13 **The firm's expected profit versus the second price segment. The parameter values are:** $r = 0.025$, $\eta = 4$, $I_0 = 10$, $\sigma = 2$, $\alpha = 15$, **and** $\beta = 1$.

Lemma 2. *Under static contracting and when the switching point is $T/2$, the optimal first segment price p_1^* is no higher than the second segment price p_2, that is, $p_1^* \le p_2^*$. The equality is achieved when the discount rate r is zero.*

The lemma below addresses the effect of demand volatility.

Lemma 3. *Under static contracting and when the switching point is $T/2$, the optimal first segment price p_1^* and the optimal second segment price p_2^* both rise in demand volatility σ.*

Figure 14 shows that both the first segment and the second segment prices increase in the demand volatility σ.

6.2. Varying Switching Points

We continue with the static contracting and examine the effect of varying switching points for the price segments. For any switching point $t' \in [0, T]$, now the firm's value function is given by

$$
\begin{aligned}
V^{\mathrm{p}} = & \sup_{p_1, p_2, b_0} E\left[\int_0^{t'} e^{-rt}\left(p_1\eta\beta^2 b_0 + \alpha p_1 - \alpha p_1^2 - \eta\beta^2 b_0^2\right) dt\right. \\
& \left. + \int_{t'}^{T} e^{-rt}\left(p_2\eta\beta^2 b_0 + \alpha p_2 - \alpha p_2^2 - \eta\beta^2 b_0^2\right) dt - \pi e^{-rT}\left(D_T - I_0\right)^+\right] \\
= & \sup_{p_1, p_2, b_0} (1 - e^{-rt'})(p_1\eta\beta^2 b_0 + \alpha p_1 - \alpha p_1^2 - \eta\beta^2 b_0^2) + (e^{-rt'} - e^{-rT})(p_2\eta\beta^2 b_0 + \alpha p_2 - \alpha p_2^2 - \eta\beta^2 b_0^2) \\
& - \pi e^{-rT}\int_{\bar{M}/(\sigma\sqrt{T})}^{\infty} (\sigma\sqrt{T}x - \bar{M})e^{-x^2/2}/\sqrt{2\pi}dx,
\end{aligned}
$$

where $\bar{M} = I_0 - (\alpha(1 - p_1)t' + \alpha(1 - p_2)(T - t')) - \beta^2\eta b_0 T$.

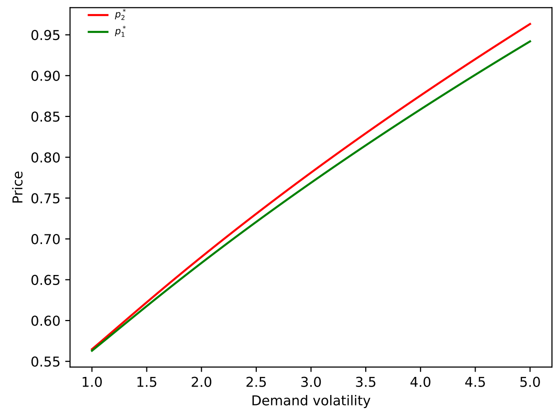


Figure 14 **Segment prices versus demand volatility. Parameter values:** $r = 0.1$**,** $\eta = 4$**,** $I_0 = 10$**,** $\alpha = 15$**,** $b_0 = 0.1$ **and** $\beta = 1$**.**

The following lemma describes how the optimal segment prices (p_1, p_2) vary against different levels of demand volatility.

LEMMA 4. *For any switching point t', the optimal first segment price p_1^* and the optimal second segment price p_2^* rise in demand volatility σ. If the discount rate r is zero then the optimal segment prices are the same, that is, $p_1^* = p_2^*$.*

Lemma 4 suggests that when a firm faces increased demand volatility, it tends to increase its selling prices. As σ increases, the firm faces a choice between attracting more demand by lowering prices or increasing marginal revenue by raising prices, and at the same, lowering the risk of overbooking penalty. However, under higher σ, even if the firm stimulates demand by lowering prices, this demand could later be offset due to high demand volatility. Therefore, using price to stimulate demand is less effective under high demand volatility than under low volatility. The second part of Lemma 4 indicates that it is optimal to use a single price segment when there is no discounting. The reason is that the optimal pricing strategy is determined by maximizing the same instantaneous function, leading to the same maximizer.

Next, motivated by the practice of the airline industry (see, e.g., Feng and Xiao 2001), we analyze the price segment policy determined by the inventory rather than the deterministic switching point as in the previous subsection.

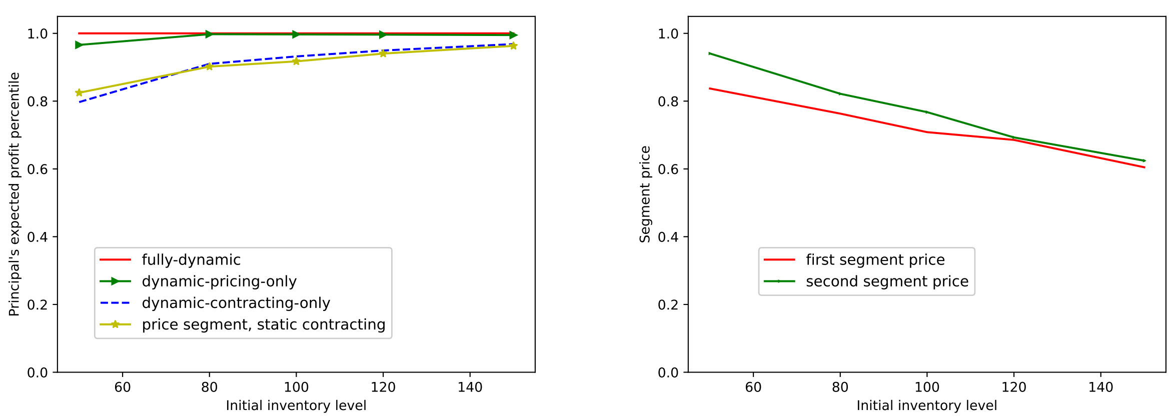
The firm uses p_1 for the first I_1 customers, where $I_1 \le I_0$. Let $\tau = \inf\{t \in [0,T] : D_t \ge I_1\}$. For $t > \tau$, the firm sets the price $p_t = p_2$ for the remaining inventory $I_0 - I_1$. The firm's problem is to decide the optimal prices p_1 and p_2, and the first segment inventory I_1. The firm's value function is given by

$$V^{\mathrm{p}} = \sup_{p_1, p_2, I_1, b_0} E\left[\int_0^{\tau} e^{-rt} p_1 dD_t + \int_{\tau}^{T} e^{-rt} p_2 dD_t - e^{-rT}\Gamma - \pi e^{-rT}\left(D_T - I_0\right)^+\right]. \tag{28}$$

As in Section 3, we normalize the agent's participation level ρ_a to zero.

Proposition 10. *Under the price segment policy, the firm's expected profit is lower than under the dynamic pricing strategy.*

We numerically present the pricing policy under the airline parameter values in Section 5. Figure 15(a) shows the dynamic-pricing-only strategy outperforms the price segment policy (under static contracts). On average, the difference between the profits of these strategies is about 10%. This differences falls in the initial inventory level I_0. Figure 15(b) shows the optimal second segment price is lower than that for the first segment. First the firm aims to boost the demand with low price p_1. Once the sales are secured at some level, the firm slows down the demand process and rises the marginal profit with high price p_2. As the inventory level rises, the two segment prices converge. With high inventory level I_0, there is no overbooking penalty, and therefore, the price difference is due to the discounting. Since the discount rate is small in the airline case, the optimal segment prices are almost the same with high initial inventory levels.



(a) **The firm's expected profit percentile versus initial inventory level.**

(b) **Segment prices versus initial inventory level.**

Figure 15 The firm's expected profit percentile and the optimal segment prices versus the initial inventory level (I_0). All the parameter values are the same as in Figure 8 except that we vary I_0 between 50 and 180.

7. Concluding Remarks

How to extract value from limited and perishable inventory is a prominent research area in the fields of operations management and marketing. Yet, the two fields have taken drastically different approaches: in the operations management (esp. dynamic pricing) literature, motivated by the problem of selling airline tickets and hotel rooms, the emphasis has been on dynamic pricing without considering agency issues; in the marketing (esp. salesforce compensation) literature, by contrast, the emphasis has been on managing agency issues under an exogenous price. Methodologically, the operations management literature rarely models unobservable actions undertaken by sales agents, whereas the marketing literature focuses on single-period models in which all consumers arrive simultaneously.

We bridge the two fields by investigating incentive design and pricing jointly using a continuous-time principal-agent model with inventory and pricing considerations. We show that under dynamic pricing, a static incentive scheme helps the firm reap nearly all the benefits of the optimal dynamic incentive scheme when inventory levels are low and the demand is elastic. This finding is consistent with the observation that airlines employ dynamic pricing but rarely offer agents dynamic contracts. By contrast, using a static pricing and incentive structure results in a significant loss of efficiency. Our additional analysis demonstrates the value of managing sales incentives and pricing jointly. We demonstrate that, outside of the airline industry, the value of incentive design can outweigh the value of dynamic pricing in industries where personalized selling strategies generate more sales. Static sales incentives, particularly when the firm lacks pricing flexibility, can result in a significant loss of efficiency.

Implementing a fully dynamic strategy may prove difficult in practice. For this reason, we compare two partially dynamic strategies in which the firm engages in either dynamic pricing or dynamic contracting but not both. Among other findings, we show that when inventory levels are high, the dynamic-contracting-only strategy outperforms the dynamic-pricing-only strategy; however, when inventory levels are low, the dynamic-pricing-only strategy outperforms the dynamic-contracting-only strategy across a broader range of parameters. In addition to these partially dynamic strategies, we analyze a case in which the firm operates on price segments and use our analytical framework to gain insight into the firm's choice of price level and length of each segment.

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Appendices to "Incentive Design and Pricing under Limited Supply"

A: Technical Proofs

In preparation for the proof of Lemma 1, we justify the specific form of the agent contract in (4). Let us first only consider incentive design under constant price as in Section 3. Let us consider the process of the agent's value function V_t^{a} that is Markov and smooth with $V_t^{\mathrm{a}} \in C^{1,2}([0,T] \times \mathbb{R})$ in the sense of functional differentiation in Dupire (2009). Then, we have

$$dV_t^{\mathrm{a}} = \partial_t V_t^{\mathrm{a}} dt + Z_t dD_t + \tfrac{1}{2}\partial_{DD} V_t^{\mathrm{a}} d\langle D\rangle_t, \tag{29}$$

where, by Itô's formula, the process $Z_t = \partial_D V_t^{\mathrm{a}}(t, D_t)$, and ∂_D and ∂_{DD} are the first and second partial derivatives with respect to the demand process D. Thus, Z_t represents the sensitivity of the agent's value function with respect to the demand process, and it is the key to inducing sales effort.

By the martingale optimality principle and (3), the process $V_t^{\mathrm{a}} e^{-rt} - \int_0^t e^{-rs} c(A_s) ds$ is a supermartingale for all admissible control processes, and it is a martingale for the optimal admissible control process provided that the optimizer exists. Then, by Itô's formula, we get

$$\begin{aligned}
& d\left[V_t^{\mathrm{a}} e^{-rt} - \int_0^t e^{-rs} c(A_s) ds\right] \\
&= -re^{-rt} V_t^{\mathrm{a}} dt + e^{-rt} dV_t^{\mathrm{a}} - e^{-rt} c(A_t) dt \\
&= -re^{-rt} V_t^{\mathrm{a}} dt + e^{-rt}\left[\partial_t V_t^{\mathrm{a}} dt + Z_t dD_t + \tfrac{1}{2}\partial_{DD} V_t^{\mathrm{a}} d\langle D\rangle_t\right] - e^{-rt} c(A_t) dt \\
&= e^{-rt}\left[-rV_t^{\mathrm{a}} + \partial_t V_t^{\mathrm{a}} + \tfrac{1}{2}\partial_{DD} V_t^{\mathrm{a}} \sigma^2 + Z_t \cdot (A_t + a) - c(A_t)\right] dt + Z_t \sigma dB_t.
\end{aligned}$$

Using the fact that the drift term of a martingale vanishes, we get the following path-dependent Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = -rV_t^{\mathrm{a}} + \partial_t V_t^{\mathrm{a}} + \tfrac{1}{2}\partial_{DD} V_t^{\mathrm{a}} \sigma^2 + H_{\mathrm{a}}(Z_t), \tag{30}$$

where we introduced the Hamiltonian function for the agent's problem:

$$H_{\mathrm{a}}(z) \equiv \max_{A \in \mathbb{R}} h_{\mathrm{a}}(z, A) = \frac{\eta z^2}{2} + za, \tag{31}$$

where $h_{\mathrm{a}}(z, A) = z \cdot (A + a) - c(A)$ for all $z \in \mathbb{R}$ and $c(A) = \frac{A^2}{2\eta}$. The maximum is attained at $\hat{A}(z) = \eta z$. Note V^{a} in (2) is the agent's value function at $t = 0$. That is, $V^{\mathrm{a}} = V_0^{\mathrm{a}}$, where the value function of the agent, V_t^{a}, is the solution of HJB equation (30) and the optimal effort of the agent is given by $\hat{A}(Z_t) = \eta Z_t$.

Substituting (30) into (29) gives

$$\begin{aligned}
dV_t^{\mathrm{a}} &= \partial_t V_t^{\mathrm{a}} dt + Z_t dD_t + \tfrac{1}{2}\partial_{DD} V_t^{\mathrm{a}} d\langle D\rangle_t \\
&= \left[rV_t^{\mathrm{a}} - H_{\mathrm{a}}(Z_t)\right] dt + Z_t dD_t.
\end{aligned} \tag{32}$$

By the definition of V_t^{a} in (3) and the direct integration of (32), we get

$$\xi_T = V_T^{\mathrm{a}} = M_0 + \int_0^T \left(Z_t dD_t + \left[rV_t^{\mathrm{a}} - H_{\mathrm{a}}(Z_t)\right] dt\right), \tag{33}$$

where $M_0 = V_0^{\mathrm{a}}$. The form of contract in (33) is the same as the agent's contract in (4). The constant term Y_0 in (4) is equivalent to M_0 in (33). The process Y_t in (4) delivers the agent's continuation utility V_t. In other words, the process Y keeps track of the agent's future utility, which is often referred to as the "promised utility" in the literature (see, e.g., Ljungqvist and Sargent 2004).

Proof of Proposition 1. By (4), we get

$$\begin{aligned} de^{-rt}Y_t &= -re^{-rt}Y_t dt + e^{-rt}Z_t dD_t - e^{-rt}H_{\mathrm{a}}(Z_t)dt + e^{-rt}rY_t dt \\ &= e^{-rt}Z_t dD_t - e^{-rt}H_{\mathrm{a}}(Z_t)dt. \end{aligned} \tag{34}$$

From this and equations (3) and (4), the agent's problem can be written as:

$$\begin{aligned} V^{\mathrm{a}} &= \sup_{\{A_t\}} E\left[Y_0 + \int_0^T e^{-rt}Z_t dD_t - e^{-rt}H_{\mathrm{a}}(Z_t)dt - \int_0^T e^{-rt}c(A_t)dt\right] \\ &= \sup_{\{A_t\}} E\left[Y_0 + \int_0^T e^{-rt}\left(h_{\mathrm{a}}(A_t, Z_t) - H_{\mathrm{a}}(Z_t)\right)dt\right] \le Y_0, \end{aligned} \tag{35}$$

where $h_{\mathrm{a}}(Z_t, A_t) = Z_t \cdot (A_t + a) - c(A_t)$ and the expectation is taken under the demand dynamics (1). The equality in (35) is obtained when $A_t = \hat{A}_t$ for the optimizer of the agent's Hamiltonian function. For all $t \in [0, T]$, by (34), we have

$$\begin{aligned} V_t^{\mathrm{a}} &= \sup_{\{A_s\}_{s\in[t,T)}} E\left[e^{-r(T-t)}Y_T - \int_t^T e^{-r(s-t)}c(A_s)ds | D_t\right] \\ &= \sup_{\{A_s\}_{s\in[t,T)}} E\left[Y_t + \int_t^T e^{-r(s-t)}Z_s dD_s - e^{-r(s-t)}H_{\mathrm{a}}(Z_s)dt - \int_t^T e^{-r(s-t)}c(A_s)d_s | D_t\right] \\ &= \sup_{\{A_t\}_{s\in[t,T)}} E\left[Y_t + \int_t^T e^{-r(s-t)}\left(h_{\mathrm{a}}(A_s, Z_s) - H_{\mathrm{a}}(Z_s)\right)ds | D_t\right] \le Y_t. \end{aligned} \tag{36}$$

Therefore, for any $t \in [0, T]$, $V_t^{\mathrm{a}} \le Y_t$ and equality is achieved when the agent's effort is $A_t = \hat{A}_t$, i.e., it optimizes (5). *Q.E.D.*

Proof of Theorem 1. The proof follows from the arguments that precede Theorem 1. *Q.E.D.*

Proof of Lemma 1. Following (36) and the optimal agent's effort in Proposition 1, we have $V_t^{\mathrm{a}} = Y_t$. By the process of Y in (4), the agent's continuation utility V_t^{a} evolves according to

$$dV_t^{\mathrm{a}} = \left[rV_t^{\mathrm{a}} - H_{\mathrm{a}}(Z_t)\right]dt + Z_t dD_t,$$

which completes the proof. *Q.E.D.*

Proof of Proposition 2. For any process $\{Z_s\}_{s\in[t,T]}$ and overbooking parameter $\pi > 1$, by (4) and (7), we denote, the principal's value function under a given incentive process, Z, and overbooking-penalty parameter, π, is

$$U_t^{\mathrm{p}}(\{Z_s\}_{s\in[t,T]}, c) = E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s - \eta Z_s^2/2\right)ds - (1+c)e^{-r(T-t)}\left(D_T - I_0\right)^+\right], \tag{37}$$

where the expectation is taken under the demand dynamics (1) with the agent's optimal effort $\hat{A}_t(Z_t)$. We denote $c := \pi - 1$ and $c > 0$. Then, for any $c \ge c'$, let $\{Z_s^{*c}\}_{s\in[t,T]}$ be the optimizer for the firm's expected profit function with parameter c, $\hat{V}_t^{\mathrm{p}}(c)=\sup_{\{Z_s\}_{s\in[t,T]}} U_t^{\mathrm{p}}(\{Z_s\}_{s\in[t,T]},c)$ and $\{Z_s^{*c'}\}_{s\in[t,T]}$ be the optimizer for $\hat{V}_t^{\mathrm{p}}(c')=\sup_{\{Z_s\}_{s\in[t,T]}} U_t^{\mathrm{p}}(\{Z_s\}_{s\in[t,T]},c')$. Next, we have the following:

$$\begin{aligned}
&\hat{V}_t^{\mathrm{p}}(c)=U_t^{\mathrm{p}}(\{Z_s^{*c}\}_{s\in[t,T]},c)=E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s^{*c}-\eta Z_s^{*c2}/2\right)ds-(1+c)e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\
&\le U_t^{\mathrm{p}}(\{Z_s^{*c}\}_{s\in[t,T]},c')=E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s^{*c}-\eta Z_s^{*c2}/2\right)ds-(1+c')e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\
&\le U_t^{\mathrm{p}}(\{Z_s^{*c'}\}_{s\in[t,T]},c')=E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s^{*c'}-\eta Z_s^{*c'2}/2\right)ds-(1+c')e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\
&=\hat{V}_t^{\mathrm{p}}(c')=\sup_{\{Z_s\}_{s\in[t,T]}} U_t^{\mathrm{p}}((Z_s)_{s\in[t,T]},c')
\end{aligned}$$

for any $c\ge c'$. Let us denote $V_t^{\mathrm{p}}(\pi):=\hat{V}_t^{\mathrm{p}}(c)$ for all π and $\pi=1+c$. Therefore, it is straightforward to obtain that for any $\pi\ge\pi'$, we have $V_t^{\mathrm{p}}(\pi)\le V_t^{\mathrm{p}}(\pi')$.

The proof of part (ii) of the proposition follows an approach similar to that of part (i) and is omitted for brevity. *Q.E.D.*

Proof of Proposition 3. To avoid abuse of notation, we denote $D_t^{s,d,\boldsymbol{z}}$ as the demand quantity at time t with the initial demand d, starting from time s controlled by the incentive process $\boldsymbol{z}$. By (37), the firm's expected profit function at time t is given by

$$V_t^{\mathrm{p}}(d)=\sup_{\boldsymbol{z}=\{Z_s\}_{s\in[t,T]}} E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s-\eta Z_s^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T^{t,d,\boldsymbol{z}}-I_0\right)^+|d\right].$$

For any $\lambda\in[0,1]$ and initial demand quantities d_1 and d_2 at time t, and any incentive processes $\boldsymbol{z_1}=\{Z_{1,s}\}_{s\in[t,T]}$ and $\boldsymbol{z_2}=\{Z_{2,s}\}_{s\in[t,T]}$, we have

$$\begin{aligned}
&V_t^{\mathrm{p}}(\lambda d_1+(1-\lambda)d_2)\ge E_t\left[\int_t^T e^{-r(s-t)}\left(\eta\left[\lambda Z_{1,s}+(1-\lambda)Z_{2,s}\right]-\eta\left[\lambda Z_{1,s}+(1-\lambda)Z_{2,s}\right]^2/2\right)ds\right.\\
&\left.-\pi e^{-r(T-t)}\left(D_T^{t,\lambda d_1+(1-\lambda)d_2,\lambda\boldsymbol{z_1}+(1-\lambda)\boldsymbol{z_2}}-I_0\right)^+|d=\lambda d_1+(1-\lambda)d_2\right]\\
&\ge\lambda E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_{1,s}-\eta Z_{1,s}^2/2\right)ds+(1-\lambda)\int_t^T e^{-r(s-t)}\left(\eta Z_{2,s}-\eta Z_{2,s}^2/2\right)ds\right]\\
&-E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,\lambda d_1+(1-\lambda)d_2,\lambda\boldsymbol{z_1}+(1-\lambda)\boldsymbol{z_2}}-I_0\right)^+|d=\lambda d_1+(1-\lambda)d_2\right]\\
&\ge\lambda E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_{1,s}-\eta Z_{1,s}^2/2\right)ds+(1-\lambda)\int_t^T e^{-r(s-t)}\left(\eta Z_{2,s}-\eta Z_{2,s}^2/2\right)ds\right]\\
&-\lambda E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,d_1,\boldsymbol{z_1}}-I_0\right)^+|d=d_1\right]-(1-\lambda)E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,d_2,\boldsymbol{z_2}}-I_0\right)^+|d=d_2\right]\\
&\ge\lambda E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_{1,s}-\eta Z_{1,s}^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T^{t,d_1,\boldsymbol{z_1}}-I_0\right)^+|d=d_1\right]\\
&+(1-\lambda)E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_{2,s}-\eta Z_{2,s}^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T^{t,d_2,\boldsymbol{z_2}}-I_0\right)^+|d=d_2\right].
\end{aligned}\tag{38}$$

Maximizing the right-hand side over the incentive process $\boldsymbol{z_1}$ and $\boldsymbol{z_2}$, the above inequality in (38) can be rewritten as

$$V_t^{\mathrm{p}}(\lambda d_1+(1-\lambda)d_2)\ge\lambda V_t^{\mathrm{p}}(d_1)+(1-\lambda)V_t^{\mathrm{p}}(d_2).\tag{39}$$

The concavity is obtained by (39). *Q.E.D.*

PROOF OF PROPOSITION 4. First, the HJB equation in (10) gives the optimal incentive variable Z_t^* by

$$\sup_{Z_t}\left(\eta Z_t - \eta Z_t^2/2\right) + \frac{\partial V_t^{\mathrm{p}}}{\partial D}\eta Z_t.$$

Then, by the first-order condition, we obtain $Z_t^* = 1 + \partial V_t^{\mathrm{p}}/\partial D_t$. Next, by taking the first order derivative with respect to D_t, we get

$$\partial Z_t^*/\partial D_t = \partial^2 V_t^{\mathrm{p}}/\partial D_t^2. \tag{40}$$

By Proposition 3 and (40), the concavity of V_t^{p} gives $\partial Z_t^*/\partial D_t \le 0$. Therefore, the optimal incentive Z_t^* is decreasing in D_t.

Then, by substituting $Z_t^* = 1 + \partial V_t^{\mathrm{p}}/\partial D_t$ into (10), we obtain

$$0 = \eta\left(1 + \frac{\partial V^{\mathrm{p}}}{\partial D}\right)^2/2 - rV^{\mathrm{p}} + \frac{\partial V^{\mathrm{p}}}{\partial t} + \frac{1}{2}\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2}\sigma^2. \tag{41}$$

Next, we take the first order derivative with respect to the demand quantity D for equation (41), which yields

$$0 = \eta\left(1 + \frac{\partial V^{\mathrm{p}}}{\partial D}\right)\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2} - r\frac{\partial V^{\mathrm{p}}}{\partial D} + \frac{\partial^2 V^{\mathrm{p}}}{\partial t\partial D} + \frac{1}{2}\frac{\partial^3 V^{\mathrm{p}}}{\partial D^3}\sigma^2. \tag{42}$$

Let $J = \frac{\partial V^{\mathrm{p}}}{\partial D}$, the above equation can be written as

$$0 = \eta(1+J)\frac{\partial J}{\partial D} - rJ + \frac{\partial J}{\partial t} + \frac{1}{2}\frac{\partial^2 J}{\partial D^2}\sigma^2. \tag{43}$$

Next, we define $c \triangleq \pi - 1$ and denote $J(c) := J(\pi)$ for any c. We take the first order derivative with respect to the parameter c for the equation (43), which gives:

$$0 = \eta\frac{\partial J}{\partial c}\frac{\partial J}{\partial D} + \eta(1+J)\frac{\partial^2 J}{\partial D\partial c} - r\frac{\partial J}{\partial c} + \frac{\partial^2 J}{\partial t\partial c} + \frac{1}{2}\frac{\partial^3 J}{\partial D^2\partial c}\sigma^2. \tag{44}$$

Then, by the Ito formula, we have

$$\begin{aligned}
d\left(\frac{\partial J}{\partial c}\right) &= \frac{\partial^2 J}{\partial c\partial t}dt + \frac{\partial^2 J}{\partial c\partial D}dD + \frac{1}{2}\frac{\partial^3 J}{\partial c\partial D^2}d\langle D\rangle \\
&= \frac{\partial^2 J}{\partial c\partial t}dt + \frac{\partial^2 J}{\partial c\partial D}\left[\eta Z_t^* dt + \sigma dB_t\right] + \frac{1}{2}\frac{\partial^3 J}{\partial c\partial D^2}\sigma^2 dt \\
&= \frac{\partial^2 J}{\partial c\partial t}dt + \frac{\partial^2 J}{\partial c\partial D}\left[\eta(1+J)dt + \sigma dB_t\right] + \frac{1}{2}\frac{\partial^3 J}{\partial c\partial D^2}\sigma^2 dt \\
&= \left[\frac{\partial^2 J}{\partial c\partial t} + \frac{\partial^2 J}{\partial c\partial D}\eta(1+J) + \frac{1}{2}\frac{\partial^3 J}{\partial c\partial D^2}\sigma^2\right]dt + \frac{\partial^2 J}{\partial c\partial D}\sigma dB_t \\
&= \frac{\partial J}{\partial c}\left[r - \eta\frac{\partial J}{\partial D}\right]dt + \frac{\partial^2 J}{\partial c\partial D}\sigma dB_t.
\end{aligned} \tag{45}$$

The last equality is obtained by substituting (44) into the above drift term. Next, by taking the expectations on both sides, we obtain

$$E\left[\frac{\partial J}{\partial c}(t, D_t)\right] = \frac{\partial J}{\partial c}(t-dt, D_{t-dt}) + \frac{\partial J}{\partial c}(t-dt, D_{t-dt})\left[r - \eta\frac{\partial J}{\partial D}(t-dt, D_{t-dt})\right]dt, \tag{46}$$

where dt is an infinitesimal time interval. Because V^{p} is concave, we have $r - \eta\frac{\partial J}{\partial D} = r - \eta\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2} \ge 0$. Furthermore, note $V_T^{\mathrm{p}} = -(1+c)(D_T - I_0)^+$ and $J = \frac{\partial V^{\mathrm{p}}}{\partial D}$, $\frac{\partial J}{\partial c}(T, D_T)$ is non-positive when it exists. By (46), $E\left[\frac{\partial J}{\partial c}(T, D_T)\right] = \frac{\partial J}{\partial c}(T-dt, D_{T-dt}) + \frac{\partial J}{\partial c}(T-dt, D_{T-dt})\left[r - \eta\frac{\partial J}{\partial D}(T-dt, D_{T-dt})\right]dt$. Because $\frac{\partial J}{\partial c}(T, D_T) \le 0$

and $r-\eta\frac{\partial J}{\partial D}\geq 0$, we must have $\frac{\partial J}{\partial c}(T-dt, D_{T-dt})\leq 0$. Then, by (46) and induction , we can have $\frac{\partial J}{\partial c}(t,D_t)\leq 0$, and therefore, $\frac{\partial Z_t^*}{\partial c}=\frac{\partial^2 V^{\mathrm{P}}}{\partial D\partial c}=\frac{\partial J}{\partial c}\leq 0$. Thus, the optimal incentive Z_t^* falls in the parameter c, which means that Z_t^* also falls in the overbooking penalty parameter π. *Q.E.D.*

Proof of Theorem 2. The proof follows from the arguments that precede Theorem 2. *Q.E.D.*

Proof of Proposition 5. For any processes $\{Z_s\}_{s\in[t,T]}$, $\{p_s\}_{s\in[t,T]}$ and overbooking parameter π, by (16), we define

$$\begin{aligned}&U_t^{\mathrm{P}}(\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]},\pi)\\&=E_t\left[\int_t^T e^{-r(s-t)}\left(p_s\eta\beta^2 Z_s+\alpha p_s-\alpha p_s^2-\eta\beta^2 Z_s^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T-I_0\right)^+\right],\end{aligned}$$

where the expectation is taken under the dynamics (11) with the agent's optimal effort $\hat{A}_t(Z_t)$ in (13).

For any $\pi\geq\pi'$, let $\{Z_s^{*\pi}\}_{s\in[t,T]}$ and $\{p_s^{*\pi}\}_{s\in[t,T]}$ be the optimizer of the firm's expected profit function with parameter π, $V_t^{\mathrm{P}}(\pi)=\sup_{\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]}}U_t^{\mathrm{P}}\left(\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]},\pi\right)$. Further, let $\{Z_s^{*\pi'}\}_{s\in[t,T]}$ and $\{p_s^{*\pi'}\}_{s\in[t,T]}$ be the optimizer of $V_t^{\mathrm{P}}(\pi')=\sup_{\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]}}U_t^{\mathrm{P}}\left(\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]},\pi'\right)$.

It is straightforward to show

$$\begin{aligned}&V_t^{\mathrm{P}}(\pi)=U_t^{\mathrm{P}}(\{Z_s^{*\pi}\}_{s\in[t,T]},\{p_s^{*\pi}\}_{s\in[t,T]},\pi)\\&=E_t\left[\int_t^T e^{-r(s-t)}\left(p_s^{*\pi}\eta\beta^2 Z_s^{*\pi}+\alpha p_s^{*\pi}-\alpha p_s^{*\pi 2}-\eta\beta^2 Z_s^{*\pi 2}/2\right)ds-\pi e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\&\leq U_t^{\mathrm{P}}(\{Z_s^{*\pi}\}_{s\in[t,T]},\{p_s^{*\pi}\}_{s\in[t,T]},\pi')\\&=E_t\left[\int_t^T e^{-r(s-t)}\left(p_s^{*\pi}\eta\beta^2 Z_s^{*\pi}+\alpha p_s^{*\pi}-\alpha p_s^{*\pi 2}-\eta\beta^2 Z_s^{*\pi 2}/2\right)ds-\pi' e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\&\leq U_t^{\mathrm{P}}(\{Z_s^{*\pi'}\}_{s\in[t,T]},(\{p_s^{*\pi'}\}_{s\in[t,T]},\pi')\\&=E_t\left[\int_t^T e^{-r(s-t)}\left(p_s^{*\pi'}\eta\beta^2 Z_s^{*\pi'}+\alpha p_s^{*\pi'}-\alpha p_s^{*\pi'2}-\eta\beta^2 Z_s^{*\pi'2}/2\right)ds-\pi' e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\&=V_t^{\mathrm{P}}(\pi').\end{aligned}$$

Therefore, we obtained that $V_t^{\mathrm{P}}(\pi)\leq V_t^{\mathrm{P}}(\pi')$ for any $\pi\geq\pi'$. *Q.E.D.*

Proof of Proposition 6. Because $\alpha>\eta\beta^2$ and thereby, $\alpha>\eta\beta^2/2$, it is straightforward to show that $p_t\eta\beta^2 Z_t+\alpha p_t-\alpha p_t^2-\eta\beta^2 Z_t/2$ is jointly concave in (Z_t,p_t). To avoid abuse of notation, we denote $D_t^{s,d,\boldsymbol{z},\boldsymbol{q}}$ as the demand quantity at time t with demand d at time s and with incentive and price processes $\boldsymbol{z}$ and $\boldsymbol{p}$. The firm's expected profit function at time t is given by:

$$\begin{aligned}V_t^{\mathrm{P}}(d)=\sup_{\boldsymbol{z}=\{Z_s\}_{s\in[t,T]},\boldsymbol{p}=\{p_s\}_{s\in[t,T]}}&E_t\left[\int_t^T e^{-r(s-t)}\left(p_s\eta\beta^2 Z_s+\alpha p_s-\alpha p_s^2-\eta\beta^2 Z_s^2/2\right)ds\right.\\&\left.-\pi e^{-r(T-t)}\left(D_T^{t,d,\boldsymbol{z},\boldsymbol{p}}-I_0\right)^+|d\right].\end{aligned}$$

For any $\lambda\in[0,1]$ and initial demand quantities d_1 and d_2, incentive processes $\boldsymbol{z_1}=\{Z_{1,s}\}_{s\in[t,T]}$ and $\boldsymbol{z_2}=\{Z_{2,s}\}_{s\in[t,T]}$, and any price processes $\boldsymbol{p_1}=\{p_{1,s}\}_{s\in[t,T]}$ and $\boldsymbol{p_2}=\{p_{2,s}\}_{s\in[t,T]}$, we have

$V_t^{\mathrm{P}}(\lambda d_1+(1-\lambda)d_2)$

$$
\begin{aligned}
&\geq E_t\left[\int_t^T e^{-r(s-t)}\Big(\left[\lambda p_{1,s}+(1-\lambda)p_{2,s}\right]\eta\beta^2\left[\lambda Z_{1,s}+(1-\lambda)Z_{2,s}\right]+\alpha\left[\lambda p_{1,s}+(1-\lambda)p_{2,s}\right]-\alpha\left[\lambda p_{1,s}+(1-\lambda)p_{2,s}\right]^2\right.\\
&\left.-\eta\beta^2\left[\lambda Z_{1,s}+(1-\lambda)Z_{2,s}\right]^2/2\Big)\,ds-\pi e^{-r(T-t)}\left(D_T^{t,\lambda d_1+(1-\lambda)d_2,\lambda \boldsymbol{z_1}+(1-\lambda)\boldsymbol{z_2},\lambda \boldsymbol{p_1}+(1-\lambda)\boldsymbol{p_2}}-I_0\right)^+|d=\lambda d_1+(1-\lambda)d_2\right]\\
&\geq \lambda E_t\left[\int_t^T e^{-r(s-t)}\left(p_{1,s}\eta\beta^2 Z_{1,s}+\alpha p_{1,s}-\alpha p_{1,s}^2-\eta\beta^2 Z_{1,s}^2/2\right)ds\right.\\
&\left.+(1-\lambda)\int_t^T e^{-r(s-t)}\left(p_{2,s}\eta\beta Z_{2,s}+\alpha p_{2,s}-\alpha p_{2,s}^2-\eta\beta^2 Z_{2,s}^2/2\right)ds\right]\\
&-E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,\lambda d_1+(1-\lambda)d_2,\lambda \boldsymbol{z_1}+(1-\lambda)\boldsymbol{z_2},\lambda \boldsymbol{p_1}+(1-\lambda)\boldsymbol{p_2}}-I_0\right)^+|d=\lambda d_1+(1-\lambda)d_2\right]\\
&\geq \lambda E_t\left[\int_t^T e^{-r(s-t)}\left(p_{1,s}\eta\beta^2 Z_{1,s}+\alpha p_{1,s}-\alpha p_{1,s}^2-\eta\beta^2 Z_{1,s}^2/2\right)ds\right]\\
&+E_t\left[(1-\lambda)\int_t^T e^{-r(s-t)}\left(p_{2,s}\eta\beta^2 Z_{2,s}+\alpha p_{2,s}-\alpha p_{2,s}^2-\eta\beta^2 Z_{2,s}^2/2\right)ds\right]\\
&-\lambda E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,d_1,\boldsymbol{z_1},\boldsymbol{p_1}}-I_0\right)^+|d=d_1\right]-(1-\lambda)E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,d_2,\boldsymbol{z_2},\boldsymbol{p_2}}-I_0\right)^+|d=d_2\right]\\
&\geq \lambda E_t\left[\int_t^T e^{-r(s-t)}\left(p_{1,s}\eta\beta^2 Z_{1,s}+\alpha p_{1,s}-\alpha p_{1,s}^2-\eta\beta^2 Z_{1,s}^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T^{t,d_1,\boldsymbol{z_1},\boldsymbol{p_1}}-I_0\right)^+|d=d_1\right]\\
&+(1-\lambda)E_t\left[\int_t^T e^{-r(s-t)}\left(p_{2,s}\eta\beta^2 Z_{2,s}+\alpha p_{2,s}-\alpha p_{2,s}^2-\eta\beta^2 Z_{2,s}^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T^{t,d_2,\boldsymbol{z_2},\boldsymbol{p_2}}-I_0\right)^+|d=d_2\right].
\end{aligned}
\tag{47}
$$

Maximizing the right-hand side over the incentive process and pricing process, $\boldsymbol{z_1}, \boldsymbol{p_1}$, $\boldsymbol{z_2}$, and $\boldsymbol{p_2}$, the above inequality in (47) can be rewritten as

$$
V_t^{\mathrm{p}}(\lambda d_1+(1-\lambda)d_2)\geq \lambda V_t^{\mathrm{p}}(d_1)+(1-\lambda)V_t^{\mathrm{p}}(d_2), \tag{48}
$$

which means $V_t^{\mathrm{p}}(d)$ is concave in d. *Q.E.D.*

Proof of Proposition 7. The proof consists of two parts:

(i) The optimal incentive variable and price, that is, Z_t^* and p_t^*, are given by the equation in (19). Because $\alpha>\eta\beta^2$, $(p_t\eta\beta^2 Z_t+\alpha p_t-\alpha p_t^2-\eta\beta^2 Z_t^2/2)+\frac{\partial V_t^{\mathrm{p}}}{\partial D}(\eta\beta^2 Z_t-\alpha p_t)$ is jointly concave in (p_t, Z_t) and Proposition 6 gives the concavity of the principal's value function; that is, $\frac{\partial^2 V_t^{\mathrm{p}}}{\partial D^2}\leq 0$. Then, the first-order condition with respect to p_t and Z_t gives

$$
(\eta\beta^2 Z_t+\alpha-2\alpha p_t)-\frac{\partial V_t^{\mathrm{p}}}{\partial D}\alpha=0, \qquad (p_t-Z_t)+\frac{\partial V_t^{\mathrm{p}}}{\partial D}=0.
$$

These equations give the optimal price and incentive variable $p_t^*=\frac{\frac{\partial V_t^{\mathrm{p}}}{\partial D}(\eta\beta^2-\alpha)+\alpha}{2\alpha-\eta\beta^2}$ and $Z_t^*=\frac{\frac{\partial V_t^{\mathrm{p}}}{\partial D}\alpha+\alpha}{2\alpha-\eta\beta^2}$.

(ii) By the optimal price and incentive variable in (i) and (66), we get:

$$
0=\frac{\frac{1}{2}\alpha^2\left(1+\frac{\partial V^{\mathrm{p}}}{\partial D}\right)^2}{2\alpha-\eta\beta^2}-rV^{\mathrm{p}}+\frac{\partial V^{\mathrm{p}}}{\partial t}+\frac{1}{2}\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2}\sigma^2. \tag{49}
$$

Then, we take the first order derivative with respect to the demand quantity D, which gives

$$
0=\frac{\alpha^2\left(1+\frac{\partial V^{\mathrm{p}}}{\partial D}\right)\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2}}{2\alpha-\eta\beta^2}-rV^{\mathrm{p}}-r\frac{\partial V^{\mathrm{p}}}{\partial D}+\frac{\partial^2 V^{\mathrm{p}}}{\partial t\partial D}+\frac{1}{2}\frac{\partial^3 V^{\mathrm{p}}}{\partial D^3}\sigma^2. \tag{50}
$$

Let $J=\frac{\partial V^{\mathrm{p}}}{\partial D}$, the equation (50) can be rewritten as

$$
0=\frac{\alpha^2(1+J)\frac{\partial J}{\partial D}}{2\alpha-\eta\beta^2}-rJ+\frac{\partial J}{\partial t}+\frac{1}{2}\frac{\partial^2 J}{\partial D^2}\sigma^2. \tag{51}
$$

Next, we denote $c := \pi - 1$ and define $J(c) := J(\pi)$ for any c. We take the first order derivative with respect to the parameter c for the equation (51). The reformulated equation is as follows:

$$0 = \frac{\alpha^2 \frac{\partial J}{\partial c}\frac{\partial J}{\partial D}}{2\alpha - \eta\beta^2} + \frac{\alpha^2 (1+J) \frac{\partial^2 J}{\partial D \partial c}}{2\alpha - \eta\beta^2} - r\frac{\partial J}{\partial c} + \frac{\partial^2 J}{\partial t \partial c} + \frac{1}{2}\frac{\partial^3 J}{\partial D^2 \partial c}\sigma^2. \tag{52}$$

Then, by the Ito formula, we have

$$\begin{aligned}
d\left(\frac{\partial J}{\partial c}\right) &= \frac{\partial^2 J}{\partial c \partial t}dt + \frac{\partial^2 J}{\partial c \partial D}dD + \frac{1}{2}\frac{\partial^3 J}{\partial c \partial D^2}d\langle D\rangle \\
&= \frac{\partial^2 J}{\partial c \partial t}dt + \frac{\partial^2 J}{\partial c \partial D}\left[\left(\eta\beta^2 Z_t^* + \alpha(1-p_t^*)\right)dt + \sigma dB_t\right] + \frac{1}{2}\frac{\partial^3 J}{\partial c \partial D^2}\sigma^2 dt \\
&= \frac{\partial^2 J}{\partial c \partial t}dt + \frac{\partial^2 J}{\partial c \partial D}\left[\frac{(1+J)\alpha^2}{2\alpha^2 - \eta\beta^2}dt + \sigma dB_t\right] + \frac{1}{2}\frac{\partial^3 J}{\partial c \partial D^2}\sigma^2 dt \\
&= \left[\frac{\partial^2 J}{\partial c \partial t} + \frac{\partial^2 J}{\partial c \partial D}\frac{(1+J)\alpha^2}{2\alpha^2 - \eta\beta^2} + \frac{1}{2}\frac{\partial^3 J}{\partial c \partial D^2}\sigma^2\right]dt + \frac{\partial^2 J}{\partial c \partial D}\sigma dB_t \\
&= \frac{\partial J}{\partial c}\left[r - \frac{\alpha^2}{2\alpha^2 - \eta\beta^2}\frac{\partial J}{\partial D}\right]dt + \frac{\partial^2 J}{\partial c \partial D}\sigma dB_t.
\end{aligned} \tag{53}$$

The last equality is obtained by substituting (52) into the above drift term. Next, by taking the integral and expectation for both sides, we get

$$E\left[\frac{\partial J}{\partial c}(t, D_t)\right] = \frac{\partial J}{\partial c}(t-dt, D_{t-dt}) + \frac{\partial J}{\partial c}(t-dt, D_{t-dt})\left[r - \frac{\alpha^2}{2\alpha^2 - \eta\beta^2}\frac{\partial J}{\partial D}(t-dt, D_{t-dt})\right]dt, \tag{54}$$

where dt is an infinitesimal time interval. Because V^{p} is concave and $\alpha^2 > \eta\beta^2$, we have $r - \frac{\alpha^2}{2\alpha^2-\eta\beta^2}\frac{\partial J}{\partial D} = r - \frac{\alpha^2}{2\alpha^2-\eta\beta^2}\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2} \geq 0$. Furthermore, note $V^{\mathrm{p}} = -(1+c)(D_T - I_0)^+$ and $J = \frac{\partial V^{\mathrm{p}}}{\partial D}$, $\frac{\partial J}{\partial c}(T, D_T)$ is non-positive when it exists. By (54), we have $E\left[\frac{\partial J}{\partial c}(T, D_T)\right] = \frac{\partial J}{\partial c}(T-dt, D_{T-dt}) + \frac{\partial J}{\partial c}(T - dt, D_{T-dt})\left[r - \frac{\alpha^2}{2\alpha^2-\eta\beta^2}\frac{\partial J}{\partial D}(T-dt, D_{T-dt})\right]dt$. Because $r - \frac{\alpha^2}{2\alpha^2-\eta\beta^2}\frac{\partial J}{\partial D} \geq 0$ and $\frac{\partial J}{\partial c}(T, D_T) \leq 0$, we must have $\frac{\partial J}{\partial c}(T-dt, D_{T-dt}) \leq 0$. Then, by (54) and induction, we can have $\frac{\partial J}{\partial c}(t, D_t) \leq 0$, which indicates that $\frac{\partial Z_t^*}{\partial c} = \frac{\partial^2 V^{\mathrm{p}}}{\partial D \partial c} = \frac{\partial J}{\partial c} \leq 0$. Because $p_t^* = \frac{\frac{\partial V_t^{\mathrm{p}}}{\partial D}(\eta\beta^2-\alpha)+\alpha}{2\alpha-\eta\beta^2} = \frac{J(\eta\beta^2-\alpha)+\alpha}{2\alpha-\eta\beta^2}$, $\alpha^2 > \eta\beta^2$ and $Z_t^* = \frac{\frac{\partial V_t^{\mathrm{p}}}{\partial D}\alpha+\alpha}{2\alpha-\eta\beta^2} = \frac{J\alpha+\alpha}{2\alpha-\eta\beta^2}$, we have $\frac{\partial p_t^*}{\partial c} = \frac{\frac{\partial J}{\partial c}(\eta\beta^2-\alpha)}{2\alpha-\eta\beta^2} \geq 0$ and $\frac{\partial Z_t^*}{\partial c} = \frac{\alpha\frac{\partial J}{\partial c}}{2\alpha-\eta\beta^2} \leq 0$. As such, we obtain that the optimal incentive Z_t^* falls in c and the optimal price p_t^* increases in c, which means Z_t^* also falls in the overbooking penalty parameter π and p_t^* increases in π. *Q.E.D.*

Proof of Proposition 8. By (15), the firm's expected profit under the fully-dynamic ($V_{\mathrm{p}}^{\mathrm{fd}}$), dynamic-pricing-only ($V_{\mathrm{p}}^{\mathrm{dp}}$), dynamic-contracting-only ($V_{\mathrm{p}}^{\mathrm{dc}}$), and fully-static strategies ($V_{\mathrm{p}}^{\mathrm{fs}}$), respectively, are given as follows:

$$V_{\mathrm{p}}^{\mathrm{fd}} = \sup_{\{p_t\}_{t\in[0,T]},\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p_t\eta\beta^2 Z_t + \alpha p_t - \alpha p_t^2 - \eta\beta^2 Z_t^2/2\right)dt - \pi e^{-rT}(D_T - I_0)^+\right] \tag{55}$$

$$\begin{aligned}
V_{\mathrm{p}}^{\mathrm{dp}} &= \sup_{\{p_t\}_{t\in[0,T]},b_0,a_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt}p_t dD_t - e^{-rT}\left(a_0 + b_0\int_0^T e^{-r(t-T)}dD_t\right) - \pi e^{-rT}(D_T - I_0)^+\right] \\
&= \sup_{\{p_t\}_{t\in[0,T]},b_0} E\left[\int_0^T e^{-rt}\left(p_t\eta\beta^2 b_0 + \alpha p_t - \alpha p_t^2 - \eta\beta^2 b_0^2 dt\right) - \pi e^{-rT}(D_T - I_0)^+\right]
\end{aligned} \tag{56}$$

$$\begin{aligned}
V_{\mathrm{p}}^{\mathrm{dc}} &= \sup_{p,\xi_T} E\left[\int_0^T e^{-rt}p dD_t - e^{-rT}\xi_T - \pi e^{-rT}(D_T - I_0)^+\right] \\
&= \sup_{p,\{Z_t\}_{t\in[0,T]},Y_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt}p dD_t - e^{-rT}Y_T^Z - \pi e^{-rT}(D_T - I_0)^+\right]
\end{aligned}$$

$$= \sup_{p,\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 Z_t + \alpha p - \alpha p^2 - \eta\beta^2 Z_t^2/2dt\right) - \pi e^{-rT}\left(D_T - I_0\right)^+\right] \tag{57}$$

$$V_{\mathrm{p}}^{\mathrm{fs}} = \sup_{p,b_0,a_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt} p dD_t - e^{-rT}(a_0 + b_0\int_0^T e^{-r(t-T)} dD_t) - \pi e^{-rT}\left(D_T - I_0\right)^+\right]$$
$$= \sup_{p,b_0} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 b_0 + \alpha p - \alpha p^2 - \eta\beta^2 b_0^2 dt\right) - \pi e^{-rT}\left(D_T - I_0\right)^+\right], \tag{58}$$

where the static contract is given by $\Gamma = a_0 + b_0\int_0^T e^{-r(t-T)} dD_t$ and the dynamic contract ξ_T is given by (14). Here the agent's participation level is normalized to zero (i.e., $\rho_{\mathrm{a}} = 0$), and thereby, the optimal base compensations Y_0 and a_0 are equal to the normalized participation level, that is, $Y_0 = 0$ and $a_0 = 0$.

The instantaneous incentive cost is the instantaneous promised payment to induce the desired effort. The results are directly derived from (20) to (23). For example, to examine the difference between static contracting and dynamic contracting, let us first compare the firm's value functions under the dynamic-contracting-only strategy ($V_{\mathrm{p}}^{\mathrm{dc}}$) and under the fully-static strategy ($V_{\mathrm{p}}^{\mathrm{fs}}$). Under the dynamic-contracting-only strategy, by (15), the firm's value function is given by:

$$V_{\mathrm{p}}^{\mathrm{dc}} = \sup_{p,\xi_T} E\left[\int_0^T e^{-rt} p dD_t - e^{-rT}\xi_T - \pi e^{-rT}\left(D_T - I_0\right)^+\right]$$
$$= \sup_{p,\{Z_t\}_{t\in[0,T]},Y_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt} p dD_t - e^{-rT} Y_T^Z - \pi e^{-rT}\left(D_T - I_0\right)^+\right]$$
$$= \sup_{p,\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 Z_t + \alpha p - \alpha p^2 - \eta\beta^2 Z_t^2/2\right) dt - \pi e^{-rT}\left(D_T - I_0\right)^+\right], \tag{59}$$

where the agent's participation level ρ_{a} can be normalized to zero and the optimal Y_0 is just equal to the normalized participation level, that is, $Y_0 = 0$. The last equality is obtained by substituting the optimal $Y_0 = 0$, the demand dynamics in (6), and payoff process in (14) .

Under the fully-static strategy, by (15), the firm's value function, $V_{\mathrm{p}}^{\mathrm{fs}}$, is given by:

$$V_{\mathrm{p}}^{\mathrm{fs}} = \sup_{p,b_0,a_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt} p dD_t - e^{-rT}\Gamma - \pi e^{-rT}\left(D_T - I_0\right)^+\right]$$
$$= \sup_{p,b_0,a_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt} p dD_t - e^{-rT}(a_0 + b_0\int_0^T e^{-r(t-T)} dD_t) - \pi e^{-rT}\left(D_T - I_0\right)^+\right]$$
$$= \sup_{p,b_0} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 b_0 + \alpha p - \alpha p^2 - \eta\beta^2 b_0^2\right) dt - \pi e^{-rT}\left(D_T - I_0\right)^+\right]. \tag{60}$$

Here, the static contract is given by $\Gamma = a_0 + b_0\int_0^T e^{-r(t-T)} dD_t$. The base payment a_0 and incentive bonus b_0 are chosen at the initial time by the firm. The last equality is obtained by substituting the demand dynamics in (11).

Next, through (59) and (60), we can compare the firm's value under the dynamic contract and the static contract. Under the static contract, the corresponding incentive parameter b_0 is chosen by the firm at the initial time $t = 0$. For the dynamic contract, the incentive variable Z_t is adapted and dependent on the current demand level or the remaining inventory level. Dynamic contracting provides the firm with more flexibility to control the demand process. For example, when the current cumulative demand is very high, the firm can provide the agent with a lower incentive, which is intended to decrease the incentive variable Z_t and to refrain from motivating the agent to generate excessive demand. Moreover, at any time t, given the same level of incentive (i.e., $Z_t = b_0$), we compare the term $(p\eta\beta^2 Z_t + \alpha p - \alpha p^2 - \eta\beta^2 Z_t^2/2)$ in **??** under the dynamic-contracting-only strategy and $(p\eta\beta^2 b_0 + \alpha p - \alpha p^2 - \eta\beta^2 b_0^2)$ in (60) under the fully-static strategy. The differences are $\eta\beta^2 Z_t^2/2$ and $\eta\beta^2 b_0^2$. Therefore, even under the same incentive level, the instantaneous incentive cost for the firm under the dynamic-contracting-only strategy is lower than that under the fully-static strategy.

Proof of Proposition 9. We provide the proof part by part:

(i) As β approaches zero or infinity, the optimal incentive Z approaches zero. Hence, the dynamic-pricing-only strategy outperforms the dynamic-contracting-only strategy.

(ii) As α approaches zero or infinity, the optimal price p approaches zero. The problem thus simplifies to a dynamic contracting problem under limited inventory as in Section 3.

(iii) As I_0 approaches infinity, our problem converges to the first-best situation. The price and incentive become time deterministic. The firm's problem simplifies to a deterministic one, rendering both the dynamic-contracting-only and dynamic-pricing-only strategies equivalent to each other.

(iv) As σ approaches zero or infinity, the optimal incentive Z approaches zero and the price p approaches one. As σ approaches zero, our problem simplifies to a deterministic problem. Hence, both the dynamic-contracting-only and dynamic-pricing-only strategies become equivalent to each other. *Q.E.D.*

Proof of Lemma 2. The firm's value function (27) can be rewritten as:

$$\begin{aligned} V^{\mathrm{p}} = \sup_{p_1,p_2,b_0} & (1-e^{-rT/2})(p_1\eta\beta^2 b_0 + \alpha p_1 - \alpha p_1^2 - \eta\beta^2 b_0^2) + (e^{-rT/2} - e^{-rT})(p_2\eta\beta^2 b_0 + \alpha p_2 - \alpha p_2^2 - \eta\beta^2 b_0^2) \\ & - \pi e^{-rT}\left(D_T - I_0\right)^+, \end{aligned}$$

where $\bar{M} = I_0 - (\alpha(1-p_1) + \alpha(1-p_2))T/2 - \beta^2\eta b_0 T$.

By the first-order condition with respect to p_1 and p_2, we have:

$$(1-e^{-rT/2})(\eta\beta^2 b_0 + \alpha - 2\alpha p_1) + \alpha\pi e^{-rT}(1 - F(\bar{M}/(\sigma\sqrt{T})))T/2 = 0 \tag{61}$$

$$(e^{-rT/2} - e^{-rT})(\eta\beta^2 b_0 + \alpha - 2\alpha p_2) + \alpha\pi e^{-rT}(1 - F(\bar{M}/(\sigma\sqrt{T})))T/2 = 0 \tag{62}$$

Because $1 - e^{-rT/2} \ge e^{-rT/2} - e^{-rT}$ and the equality is achieved when $r = 0$, the optimal solutions p_1^* and p_2^* that solve the above two equations must satisfy $p_1^* \le p_2^*$ and the equality is obtained when $r = 0$. *Q.E.D.*

Proof of Lemma 3. By (61) and (62), we have the $p_2 = \frac{\eta\beta^2 b_0 + \alpha}{2\alpha} - \frac{(1-e^{-rT/2})}{2\alpha(e^{-rT/2} - e^{-rT})}(\eta\beta^2 b_0 + \alpha - 2\alpha p_1)$. Then we substitute p_2 into (61), we have $g(p_1,\sigma) = (1-e^{-rT/2})(\eta\beta^2 b_0 + \alpha - 2\alpha p_1) + \alpha\pi e^{-rT}(1 - F(\bar{M}/(\sigma\sqrt{T})))T/2 = 0$.

By the Implicit Function Theorem, we have $\frac{\partial p_1}{\partial\sigma} = -g_{p_1}/g_\sigma$. It is easy to check that $g_p \le 0$ and $g_\sigma > 0$. Thus, $\frac{\partial p_1}{\partial\sigma} \ge 0$. We can also have $\frac{\partial p_2}{\partial\sigma} = \frac{1-e^{-rT/2}}{e^{-rT/2} - e^{-rT}}\frac{\partial p_1}{\partial\sigma} \ge 0$. Therefore, p_1^* and p_2^* increases in the demand volatility σ. *Q.E.D.*

Proof of Lemma 4. Let us denote:

$$g_1(p_1,p_2,\sigma) = (1-e^{-rt'})(\eta\beta^2 b_0 + \alpha - 2\alpha p_1) + \alpha\pi e^{-rT}(1 - F(\bar{M}/(\sigma\sqrt{T})))t'$$

$$g_2(p_1,p_2,\sigma) = (e^{-rt'} - e^{-rT})(\eta\beta^2 b_0 + \alpha - 2\alpha p_2) + \alpha\pi e^{-rT}(1 - F(\bar{M}/(\sigma\sqrt{T})))(T - t'),$$

with $g_1(p_1^*, p_2^*, \sigma) = 0$ and $g_2(p_1^*, p_2^*, \sigma) = 0$

By the Implicit Function Theorem for a system of equations, we have

$$\begin{vmatrix} \frac{\partial g_1}{\partial p_1} & \frac{\partial g_1}{\partial p_2} \\ \frac{\partial g_2}{\partial p_1} & \frac{\partial g_2}{\partial p_2} \end{vmatrix} \neq 0$$

and with the following two equations,

$$\begin{pmatrix} \frac{\partial g_1}{\partial p_1} & \frac{\partial g_1}{\partial p_2} \\ \frac{\partial g_2}{\partial p_1} & \frac{\partial g_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} p_1'(\sigma) \\ p_2'(\sigma) \end{pmatrix} = - \begin{pmatrix} \frac{\partial g_1}{\partial \sigma} \\ \frac{\partial g_2}{\partial \sigma} \end{pmatrix},$$

we can obtain the $p_1^{*'}(\sigma) \geq 0$ and $p_2^{*'}(\sigma) \geq 0$. In other words, p_1^* and p_2^* increases in the demand volatility σ.

Next, when the time discount rate $r = 0$, consider a pricing strategy p_t as a time deterministic function by varying the price at each instant of time, it is always optimal to keep p_t to be constant over time. Let us denote the optimal price under the time deterministic pricing strategy as p^*. Since the segment policy that when $t' \in [0, t'], p_t = p_1$ and $t \in (t', T], p_t = p_2$ is a special case as the time deterministic pricing strategy. Therefore, the optimality under the segment policy achieves by letting $p_1 = p_2 = p^*$. *Q.E.D.*

B: Competition between Direct Sales Channel and Sales Channel through an Agent

In this section, we apply our analytical framework to examine the competition effect between two channels of sales – direct sales and sales through an agent. The firm uses these two channels to sell the product, as explored in Section 3. The cumulative demand quantity, D_t, for the product up to any given time $t \in [0, T]$, evolves according to

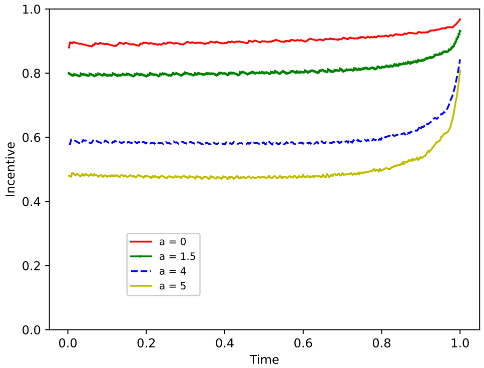
$$dD_t = (a + A_t)dt + \sigma dB_t, \tag{63}$$

where a corresponds to the firm's demand rate through its own direct sales channel, $A_t \geq 0$ is the agent's effort level, and σ is a constant diffusion term.

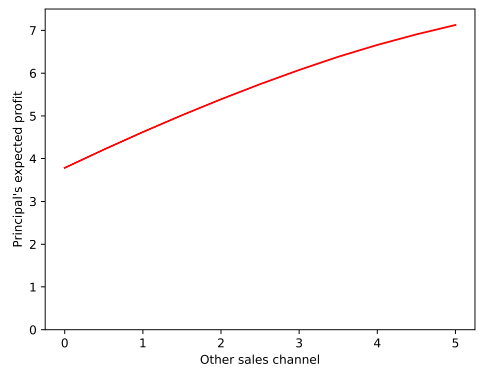
The agent's optimal effort level, given the incentive variable Z_t, is $\hat{A}_t(Z_t) = \eta Z_t$. We explore how the direct sales channel influences the agent's effort level. Up to time T, the total sales through the direct channel are aT. Therefore, the remaining inventory for the agent is $I_0 - aT$. As the direct sales channel a increases, the agent has less to sell. Equivalently, we may consider transferring the problem of increasing a to that of decreasing I_0.

From the principal's perspective, the incentive variable provided by the principal decreases with increasing direct sales a. Consequently, the expected optimal effort of the agent decreases, and thus, the expected sales, $E(\int_0^T \hat{A}_t(Z_t)dt)$ (i.e., $\int_0^T \eta Z_t dt$), also decrease with a. We numerically show how the agent's effort decreases with I_0. Similarly, we could conduct comparative statics numerically for a.

As the value of the other channel a increases, we observe from the numerical study presented in Figure 16 a decrease in the average incentive offered to the sales agents. At the same time, the firm's expected profit shows an upward trend as a increases. With a higher presence of alternative sales channels, the firm reduces its dependence on the sales agent's effort. This reduction in dependence, in turn, leads to a reduction in the cost of incentives provided to motivate these agents. Consequently, as the influence of other channels a escalates, the firm's expected profit experiences a corresponding increase.



(a) **Incentive versus time**



(b) **The firm's expected profit versus other sales channel**

Figure 16 **Optimal incentives and the firm's expected profit under different alternative sales channels** a**.** **Parameter values :** $r = 0.025$, $\eta = 8$, $I_0 = 10$, $\sigma = 2$, **and** $\pi = 2$.

C: Comparison of the Dynamic Contract with Other Contracts

In this section, we contrast our dynamic contract with other forms of contracts. We are particularly interested in exploring the differences in the principal's value, the incentive variables, and the agent's optimal effort. First, we consider a contract of the form $\xi_T^Q = B + RD_T + M(D_T - I_0)^+$, where B is the base pay, R denotes the commission rate, and M is the marginal overbook penalty with $M > R$. This contract only depends on the terminal demand D_T and also takes limited inventory into account. We can numerically search for the optimal B, R, and M for this form of contract and compare the results with our dynamic contract. Second, we examine contracts in which the firm does not compensate the agent for demand that exceeds the capacity. At time $\hat{\mathbb{T}} = \{t \geq 0 : D_t \geq I_0\}$, we set the incentive variable $Z_t = 0$ for $t \in \hat{\mathbb{T}}$. This scenario can be seen as a subset of our contract, similar to adding an additional constraint where $Z_t = 0$ when demand exceeds capacity.

Linear Penalty Contract. We start with considering the linear penalty contract that makes the agent share the overbooking penalty with the principal. We consider the contract $\xi_T^Q = e^{rT}B + \int_0^T e^{-r(t-T)} R dD_t - M(D_T - I_0)^+$, where B is the base pay, R denotes the commission rate, and M is the marginal overbook penalty with $M > R$. This contract is only dependent on the terminal demand D_T and also takes the limited inventory into account. We could numerically search for the optimal B, R, and M in this form of contract and compare it with our dynamic contract.

Corresponding to the demand process (11) and given the contract ξ_T^Q, the agent chooses optimal effort $A = \{A_t, 0 \leq t \leq T\}$ by maximizing his expected utility:

$$V^a = \sup_{\{A_t\}_{\{0 \leq t \leq T\}}} E\left[e^{-rT}\xi_T^Q - \int_0^T e^{-rt} c(A_t) ds\right],$$
$$= \sup_{\{A_t\}_{\{0 \leq t \leq T\}}} E\left[B + \int_0^T e^{-rt}\left(RA_t - c(A_t)\right) dt - Me^{-rT}(D_T - I_0)^+\right], \tag{64}$$

and without loss of generality, let $B = 0$ and $a = 0$ as in the main text.

Then, we write the agent's value function as

$$V_t^{\mathrm{a}} = \sup_{\{A_s\}_{\{t\le s\le T\}}} E\left[\int_t^T e^{-r(s-t)}\left(RA_s - A_s^2/(2\eta)\right)ds - Me^{-r(T-t)}(D_T - I_0)^+\right], \tag{65}$$

where the base pay B could be normalized to zero.

Then, we have the following HJB equation for the agent's problem:

$$\sup_{A_t}\left\{RA_t - A_t^2/(2\eta) - rV_t^{\mathrm{a}} + \frac{\partial V_t^{\mathrm{a}}}{\partial t} + \frac{\partial V_t^{\mathrm{a}}}{\partial D}A_t + \frac{1}{2}\frac{\partial^2 V_t^{\mathrm{a}}}{\partial D^2}\sigma^2\right\} = 0, \tag{66}$$

which gives the following theorem.

THEOREM 3. *The agent's optimal effort A_t satisfies*

$$\sup_{A_t}\left\{RA_t - c(A_t) + \frac{\partial V_t^{\mathrm{a}}}{\partial D}A_t\right\} \tag{67}$$

and is given by $\hat{A}_t = \eta\left(R + \frac{\partial V_t^{\mathrm{a}}}{\partial D}\right)$.

Then, the firm 's value function is given by

$$\begin{aligned} V^{\mathrm{p}} &= \sup_{\xi_T^Q} E\left[\int_0^T e^{-rt}dD_t - e^{-rT}\xi_T^Q - \pi e^{-rT}(D_T - I_0)^+\right] \\ &= \sup_{\{p_t\}_{t\in[0,T]},(R,M)} E\left[\int_0^T e^{-rt}(1-R)\left[a + \hat{A}_t\right]dt - (\pi - M)e^{-rT}(D_T - I_0)^+\right], \end{aligned} \tag{68}$$

where $R \le M \le \pi$. The firm's expected profit function at $t \in [0,T]$ is given by

$$\begin{aligned} V_t^{\mathrm{p}} &= \sup_{\xi_T^Q} E\left[\int_t^T e^{-r(s-t)}dD_t - e^{-r(T-t)}\xi_T^Q - \pi e^{-r(T-t)}(D_T - I_0)^+\right] \\ &= \sup_{(R,M)} E\left[\int_t^T e^{-r(s-t)}(1-R)\left[a + \hat{A}_s\right]ds - (\pi - M)e^{-r(T-t)}(D_T - I_0)^+\right], \end{aligned} \tag{69}$$

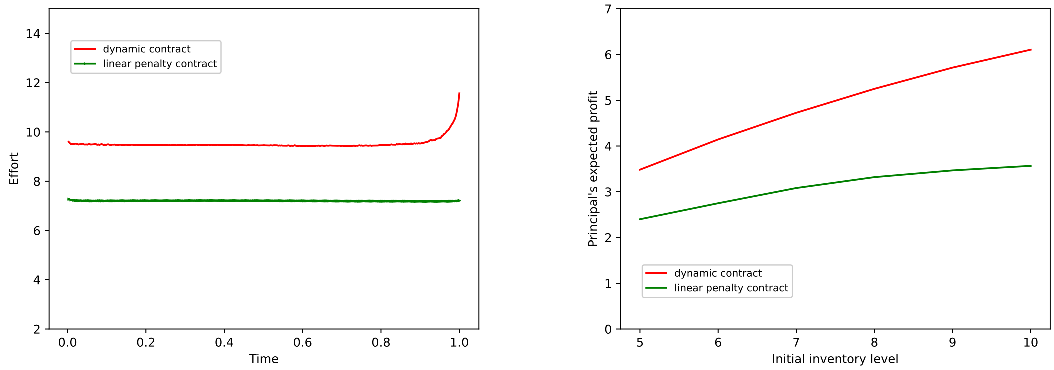
where $R \le M \le 1$ and $\hat{A}_s$ is given by Theorem 3.

THEOREM 4. *The firm's expected profit function in V_t^{p} under the dynamic contract dominates the firm's expected profit under the linear penalty contract ξ_T^Q.*

THEOREM 5. *The agent's value under the linear penalty contract ξ_T^Q weakly dominates the agent's value under our dynamic contract ξ_T.*

We now provide a numerical illustration of the agent's actions under the two contracts, detailing both the path of effort and the path of incentives under the benchmark parameters. As shown in Figure 17, the sales agent's effort under the linear penalty contract is lower. Given that the sales agent shares the penalty cost, the linear penalty contract is not as effective at inducing effort. Moreover, the principal's value under a linear penalty contract is lower than it is under our dynamic contract. As inventory levels increase, this disparity grows more pronounced, primarily because the linear penalty contract proves to be more costly per unit of demand generated. On average, the value difference between the two contracts is approximately 40%.

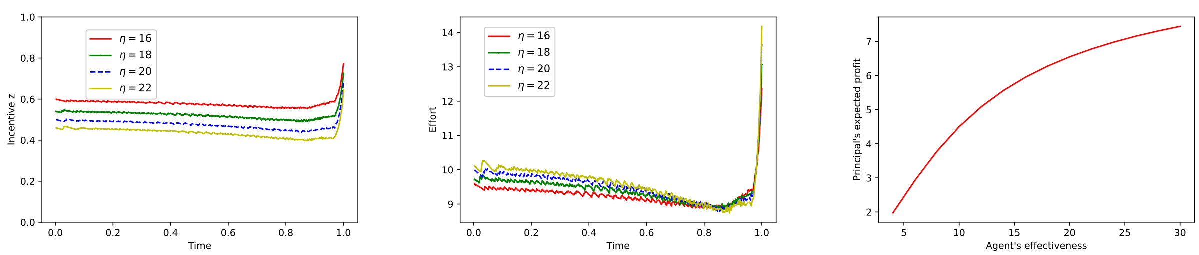
The Case of a Hard Inventory Constraint. Next, we analyze the case with a hard inventory constraint. We use numerical experiments to compare the gap between our model and that under a hard inventory



(a) **Agent's effort versus time**

(b) **The firm's expected profit versus initial inventory level**

Figure 17 **Agent's effort, and the firm's expected profit for different initial inventory I_0.**
Parameter values : $r = 0.025$, $\eta = 15$, $\sigma = 2$, $I_0 = 10$ and $\pi = 2$. The left side represents the comparison between linear penalty contract and dynamic contract for $I_0 = 10$. In this case, the optimal linear penalty contract is given by $(R, M) = (0.5, 0.6)$.



(a) **Incentive versus time**

(b) **Agent's effort versus time**

(c) **The firm's expected profit versus agent's effectiveness**

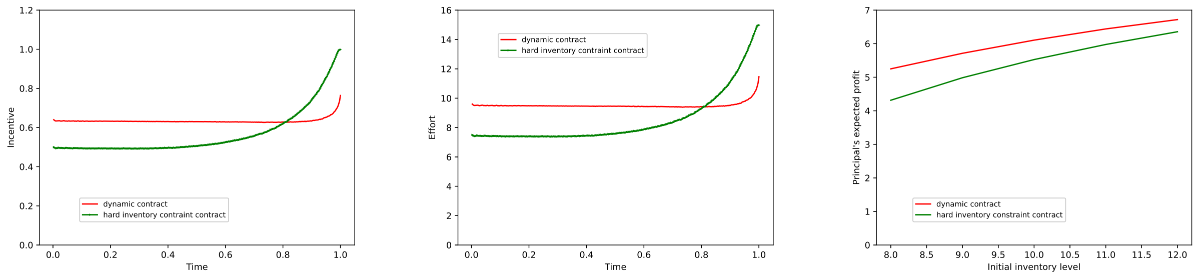
Figure 18 **Incentive, agent effort, and the firm's expected profit for different levels of salesforce effectiveness η.**
Parameter values : $r = 0.025$, $I_0 = 10$, $\sigma = 2$, and $\pi = 2$.

constraint. As shown in Figure 19, the firm's expected profit under the hard inventory constraint is higher than that under the hard inventory constraint contract. The incentive effectiveness is lower because the firm cannot do overbooking and the demand that exceeds the inventory becomes lost sales. Therefore, at the beginning, faced with ample inventory, the principal provides lower-powered incentives to smooth the increment of demand, and then uses higher-powered incentives to clear the remaining inventory at the end.

D: Digital Marketing versus Traditional Marketing

In this extension, we apply our general framework to compare and understand how digital marketing and traditional marketing vary in influencing the decision-making processes of the principal and the agent.

Digital marketing, characterized by unique features such as customer tracking over visits, comparison shopping, and knowledge of customer choice sets, provides the agent with customer visit data. Leveraging these data sources for targeted strategies, the agent can attract customers more efficiently. Consequently, the agent's effectiveness parameter, η, tends to be higher with digital marketing than with traditional marketing.



(a) **Incentive versus time** (b) **Agent's effort versus time** (c) **The firm's expected profit versus initial inventory level**

Figure 19 Incentive, agent effort, and the firm's expected profit for different initial inventory I_0**.**
Parameter values : $r = 0.025$, $\eta = 15$, $\sigma = 2$, $I_0 = 10$, **and** $\pi = 2$.

We conducted a numerical analysis to assess the impact of the effectiveness parameter on the decisions of the principal and the agent. As shown in Figure 18, the average incentive is inversely proportional to the agent's effectiveness parameter η. A higher effectiveness parameter η means the principal needs less incentive to induce the same level of effort from the agent to meet the inventory level, because the agent's optimal effort is expressed as $\hat{A}(Z_t) = \eta Z_t$. Additionally, the agent's average effort first increases then decreases in η over time. Interestingly, the principal's expected profit is directly proportional to the agent's effectiveness parameter η.

Figure 18 illustrates that the average incentive provided to the sales agent in digital marketing is smaller than that in traditional marketing under limited inventory conditions. This difference can be attributed to the lower costs and increased effectiveness of attracting customers through digital marketing. At the onset of the time horizon, when the firm has ample inventory, it prefers high sales effort to deplete the inventory. Thus, it provides a high-powered compensation plan to the more effective sales agent in the digital marketing channel, resulting in higher cumulative demand and lower remaining inventory. However, as the end of the time horizon approaches, and inventory depletes, the firm becomes wary of overbooking. In response, it offers a lower-power compensation plan to the digital marketing agent. Also note the firm's expected profit is higher in the digital marketing scenario.